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RESEARCH ARTICLE

A novel iteration algorithm for a hybrid pair of total asymptotically non-expansive single-valued and total asymptotically quasi-non-expansive multi-valued non-self mappings in Banach spaces

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Abstract: In this paper, we introduce a new mixed-type iteration scheme for finite family of total asymptotically nonexpansive single-valued and finite family of total asymptotically quasi-nonexpansive multi-valued nonself mappings. In addition, we prove weak and strong convergence theorems of the proposed iteration algorithm in Banach spaces. Our main results were obtained without any strict condition imposed on the fixed point set of the mappings, hence making our results more general in terms of application than some other existing results recently announced in literature.

Keywords: Total asymtotically nonexpansive single-valued mappings, Total asymptotically quasi-nonexpansive multivalued mapping, Mixed type iteration scheme, Uniformly convex Banach space.

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1 Introduction

Let E be a Banach space and D a nonempty closed convex subset of E. Throughout the paper, N, $CB(E), KC(E)$ *and* $\mathcal{P}(E)$ will denote the set of positive integers, the family of closed ad bounded subsets of E, the family of nonempty compact convex subsets of E and the family of proximinal subsets of E respectively.

A subset D is called a proximinal if for each $x \in E$, there exists a point $m \in D$ such that

$$d(x,m) = \inf\{\|x-y\| : y \in D\} = d(x,D). \tag{1.1}$$

It is well known that weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach spaces are proximinal.

Let $G : D \longrightarrow E$ be a sinle-valued mapping of D into E and $T : D \longrightarrow CB(E)$ be multivalued mapping of D into $CB(E)$. The set of fixed points of G and T will be denoted by $F(G) = \{x \in D : x = Gx\}$ and $F(T) = \{x \in E : x \in Tx\}$, respectively. A point x is a common fixed point of G and T if $x \in F(G) \cap F(T)$.

The first fixed point theorem for single-valued self mappings was established in 1965 by Browder [44]. He proved that if C is a bounded closed convex subset of a Hilbert space H and T is a nonexpansive mapping of C into itself, then T has a fixed point in C. Almost immediately, both Browder [45] and Gohde [46] proved that the same is true if E is a uniformly convex Banach space. Kirk [47] also proved the following theorem:

Theorem 1.1. $(\star)$ *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then F(T) is nonempty.*

After Kirk's theorem, many fixed point theorems concerning single-valued mappings have been proved in a Hilbert space or a Banach space (see, e.g., [1]-[21] and the references contained in them). In particular, Baillon and Schoneberg [48] introduced the concept of asymptotic normal structure and generalized Kirk's fixed point theorem as follows:

Theorem 1.2. $(\star\star)$ *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has asymptotic normal structure. Let T be a nonexpansive mapping of C into itself. Then F(T) is nonempty.*

In recent times. the study of fixed point of multivalued mappings has attracted the interest of a good number of well known mathematicians (see [27]-[40], and the references contained in them). Such interest, perhaps, stems from the usefulness of fixed point theory in real life applications such as in Game Theory and Market Economy, and in some other area of mathematics such as Nonsmooth Differential equations, Optimization Theory and Differential Inclusion. The following are the connection of fixed point theory of multivalued mapping and these applications:

Game Theory and Market Economy[43]. ln game theory and market economy, the existence of equilibrium was uniformly obtained by the application of fixed point theorem. More precisely, under some regularity conditions, given any game, there always exists a multivalued mapping whose fixed point coincide with equilibrium points of the game.An illustrative example of such application is the Nash equilibrium theorem [50].

Consider a game $G = (v_n, K_n)$ with N players denoted by $n = 1, 2, \cdots, N$, where $K_n \subset R^n$ is the set of possible strategies of the n^{th} player and is assumed to be nonempty, compact and convex, and $v_n : K = K_1 \times K_2 \times \cdots \times K_N \longrightarrow R$ is the game function (or payoff) of the player n and is assumed to be continuous. The player n can take individual actions represented by a vector $\sigma_n \in K_n$. All players together can take a collective action, which is a combined vector $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_N$. For each $n, \sigma \in K$ and $w_n \in K_n$, we use the following standard notations:

$$\begin{aligned} K_{-n} &= K_1 \times K_2 \times K_{n-1} \times \cdots \times K_N \\ \sigma_{-n} &= \sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \sigma_{n+1} \cdots, \sigma_N \\ (w_n, \sigma_{-n}) &= \sigma_1, \sigma_2, \cdots, \sigma_{n-1}, w_n, \sigma_{n+1} \cdots, \sigma_N \end{aligned}$$

A strategy $\bar{\sigma_n} \in K_n$ permits the n^{th} player maximise his game under the condition that the remaining players have chosen their strategies σ_{-n} if and only if

$$v_n(\bar{\sigma_n}, \sigma_{-n}) = max_{w_n \in K_n} v_n(w_n, \sigma_{-n})$$

Now, let $T_n : K_{-n} \longrightarrow 2^K$ be the multivalued mapping defined by

$$T_n(\sigma_{-n}) = Argmax_{w_n \in K_n} v_n(w_n, \sigma_{-n}), \forall \sigma_{-n} \in K_{-n}$$

Definition 1.3. *A collective action $\bar{\sigma} = (\bar{\sigma_1}, \bar{\sigma_2}, \cdots, \bar{\sigma_N}) \in K$ is called a Nash equilibrium point if, for each $n, \bar{\sigma_n}$ is the best response for the n^{th} player to the action $\bar{\sigma_n}$ made by the remaining players. That is, for each n,*

$$v_n(\bar{\sigma} = max_{w_n \in K_n} v_n(w_n, \sigma_{-n})$$

or equivalently,

$$\bar{\sigma_n} \in T_n(\bar{\sigma_{-n}})$$

This is equivalent to say that $\bar{\sigma}$ is a fixed point of the multivalued mapping $T : K \longrightarrow 2^K$ defined by

$$T(\sigma) = T_1(\sigma_{-1}) \times T_1(\sigma_{-2}) \cdots \times T_1(\sigma_{-N}) \tag{1.2}$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multivalued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by iterative methods for fixed point of multivalued mappings.

Differential Inclusion[49]. For $\Omega = (0, \pi)$, consider the following differential inclusion:

$$\begin{cases} -\dfrac{d^2u}{dt^2} \in u - \dfrac{1}{4} - \dfrac{1}{4}sgn(u-1), t \in \Omega; \\ u(0) = 0 \\ u(\pi) = 0, \end{cases} \tag{1.3}$$

where

$$sgn(x)\begin{cases} -1 & \text{if} x<0; \\ [-1,1] & \text{if} x=0 \\ 1 & \text{if} x>0 \end{cases}$$

Let $H = H_0^1(\Omega)$ be a Hilbert space and $\langle .,. \rangle_H$ be the inner product on H defined by

$$\langle u, v\rangle_H = \int_\Omega u'v'dt, \forall u, v \in H.$$

From Riesz Theorem, there exists an operator $A : H \longrightarrow H$ satisfying

$$\langle Au, v\rangle_H = \int_\Omega u'v'dt, \forall v \in H.$$

For $u \in H$, let

$$E(u) = u - \frac{1}{4} - \frac{1}{4}sgn(u-1).$$

For $w \in E(u)$, let $L_u^w : H \longrightarrow R$ be the map defined by

$$L_u^w(v) = \int_\Omega wvdt, \forall v \in H.$$

Then, L_u^w is linear and continuous on H. Therefore, using again Riesz Theorem, there exists a unique vector b_u^w such that

$$b_u^w(v) = \int_\Omega wvdt, \forall v \in H.$$

Let $B : H \longrightarrow 2^H$ be a multivalued map defined by

$$Bu = \{b_u^w : w \in E(u)\}$$

Then u is a solution of (1.2) if and only if $Au \in Bu$. Further,the operator $A : H \longrightarrow H$ is strongly monotone. If we introduce a multivalued mapping $T : H \longrightarrow 2^H$ defined by

$$Tu = u - Au - Bu, \forall u \in H,$$

Then $u \in H$ is a solution of (1) if and only if $u \in Tu$; that is u is a fixed point of T.

Optimization Problems with Constraints[49]. Let H be a real Hilbert space and $f : H \longrightarrow R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\psi : H \longrightarrow 2^H$ be a multivalued mapping. Consider the following optimization problem:

$$(P) = \begin{cases} minf(x) \\ 0 \in \psi(x). \end{cases}$$

It is known that the multivalued map, ∂f, the subdifferential of f is a maximal monotone (see [49] for details), where for $x, w \in H$,

$$w \in \partial f(x) \Leftrightarrow f(y) - f(x) \geq \langle y - x, w\rangle, \forall y \in H;$$

$$\Leftrightarrow x \in argmin(f - \langle, .w\rangle.$$

It is easily seen that, for $x \in H$, with $0 \in \psi(x), x$ is a solution of (P) if and only if $0 \in \partial f(x) \cap \psi(x)$, or equivalently

$$x \in T_1 \cap T_2,$$

with

$$T_1 = I - \partial f \qquad \text{and} \qquad T_2 = I - \psi,$$

where I is the identity map of H. Therefore, x is a solution of (P) if and only if x is a common fixed point of the multivalued mappings T_1 and T_2.

Let H be a Hausdorff metric induced by the metric d of E; that is,

$$H(A,B) = max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \tag{1.4}$$

for every $A, B \in CB(E)$.

ln the sequel, the following definitions will be needed:

Definition 1.4. *Recall that a single-valued mapping $G : D \longrightarrow D$ is called total asymptotically nonexpansive [1] if there exist sequences $\{\mu_n\}_{n\geq 1}, \{\xi_n\}_{n\geq 1} \subseteq [0,\infty) : \mu_n, \xi_n \to 0$ as $n \to \infty$ and a strictly increasing continous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that*

$$\|G^n x - G^n y\| \leq \|x-y\| + \mu_n \phi(\|x-y\|) + \xi_n, \tag{1.5}$$

for all $x, y \in D$ and $n \in N$. Observe that:

1. *If $\phi \equiv 0$, the zero operator, then (1.5) reduces to*

$$\|G^n x - G^n y\| \leq \|x-y\| + \xi_n, \tag{1.6}$$

for all $x, y \in D$ and $n \in N$, so that if D is bounded and G^N is continous for some integer $N \geq 1$, then the mapping G is called asymptotically nonexpansive-type, a class of mapping which contains the classes of both asymptotically nonexpansive mapping in the intermediate sense and nearly asymptotically nonexpansive mappings.

2. *If $\phi(t) = t$, then (1.5) reduces to*

$$\|G^n x - G^n y\| \leq (1+\mu_n)\|x-y\| + \xi_n, \tag{1.7}$$

for all $x, y \in D$ and $n \in N$, and is called generalize asymptotically nonexpansive mapping.

3. *If $\xi_n = 0$, (1.5) reduces to*

$$\|G^n x - G^n y\| \leq \|x-y\| + \mu_n \phi(\|x-y\|), \tag{1.8}$$

for all $x, y \in D$ and $n \in N$, and is called asymptotically nonexpasive mapping.

4. *If $\mu_n = 0$ and $\xi_n = 0$ for all $n \geq 1$, then the total asymptotically nonexpansive single-valued mapping coincides with a class of mapping called nonexpansive mappings.*

Alber et al [1] introdeuced the class of total asymptotically nonexpansive mappings to unify the various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove general convergence theorems applicable to all these classes.

Example 1.5. *(see [39])*
Let $E = R \times \ell$ be endowed with the norm $\|.\|_E = |.| + \|.\|_\ell$. Let K be a subset of E defined by $K = [0,1] \times B$, where B is the closed ball of ℓ_1. For all $u \in [0,1]$ and $\bar{x} = (x_1, x_2, x_3, \cdots) \in B$, define $T : K \longrightarrow K$ by

$$T(u, \bar{x}) = \begin{cases} (\frac{1}{2}, (0, \frac{|x_1|}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \cdots)) & , \, if \; u \in [0, \frac{1}{3}] \\ (0, (0, \frac{|x_1}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \cdots)) & , \, if \; u \in (\frac{1}{3}, 1] \end{cases}$$

It is shown in [39] that T is total asymptotically nonexpansive single-valued mapping but not asymptotically nonexpansive mapping.

Definition 1.6. *Recall that a multivalued mapping $T : D \longrightarrow CB(D)$ is called:*

1. *L-Lipschitizian if there exists $L > 0$ such that*

$$H(Tx, Ty) \leq L\|x - y\|, \forall x, y \in D. \tag{1.9}$$

   Note that if $L \in (0,1)$ in (1.9), then T is called a contraction; and if $L = 1$ in (1.9), then T is called nonexpansive. It is worth mentioning that the fixed points of multivalued contraction and nonexpansive mappings have been extensively researched on, and many very interesting results were obtained and extended by different researchers in recent years (see, e.g., [24], [25], [28], [29], [33], [37] and the references therein for more details).

2. *uniformly L-Lipschitizian if there exists $L > 0$ such that*

$$H(T^n x, T^n y) \leq L\|x - y\|, \forall x, y \in D, \forall n \geq 1. \tag{1.10}$$

3. *asymptotically nonexpansive if for all $x, y \in D$, there exists a squence $\{k_n\}_{n\geq 0}$ with $\lim_{n\infty} k_n = 1$ such that*

$$H(T^n x, T^n y) \leq k_n\|x - y\|, \forall n \geq 1. \tag{1.11}$$

   It is easy to see that every asymptotically nonexpansive multivalued mapping is a superclass of the class of both multivalued contraction mapping and the class of nonexpansive multivalued mapping.

4. *total asymptotically nonexpansive mapping [32] if for all $x, y \in D$ and the sequences $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1} \in [0, \infty) : \alpha_n, \beta_n \to 0$ as $n \in \infty$, there exists a strictly increasing continuous function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$ such that*

$$H(T^n x, T^n y) \leq \|x - y\| + \alpha_n \phi(\|x - y\|) + \beta_n, \forall n \geq 1. \tag{1.12}$$

Remark 1.7. *If $\phi(t) = t$, then (1.12) becomes*

$$H(T^n x, T^n y) \leq (1 + \alpha_n)\|x - y\| + \beta_n, \forall n \geq 1, \tag{1.13}$$

and is called generalize asymptotically nonexpansive multivalued mapping. Again, if $\phi(t) = t$ and $\beta_n = 0$, then (1.12) reduces to (1.11) with $k_n = 1 + \alpha_n$, while the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{k_n\}_{n\geq 1}$ still retain their properties.

Definition 1.8. *A subset K of E is said to be a retract of E if there exists a continuous mapping $R : E \longrightarrow K$ (called retraction) such that $R(v) = v$ for all $v \in K$. If, in addition R is nonexpansive, then R is said to be nonexpansive retraction of E. If $R : E \longrightarrow K$ is a retraction, then $R^2 = R$. A retract of a Hausdorff space must be a closed subset. It is well known that every closed convex subset of a uniformly convex Banach space E is a retract of E.*

The idea of approximating fixed points for multivalued contraction and nonexpansive mappings using the Hausdorff metric was initiated by Markin [24].Thereafter, various iteration schemes have been developed and used to approximate the fixed points of multivalued nonexpansive mappings in Banach spaces by different authors (see, e.g., [26]-[37] and the reference therein); and their painstaking efforts have led to interesting results in the study fixed point theory with real applications in convex optimization, control theory, economics, differential inclusion and related topics.

ln 1974, Lim [28], using the Edelstein's method of asymtotic center (see [31] for details), proved that every multivalued nonexpansive self mapping $T : E \to K(E)$ has a fixed point, where E is a nonempty bounded closed convex subset of a uniformly convex Banach space. ln 1998, Kirk and Massa [27] extended Lim's theorem to assure the existence of fixed point of multivalued nonexpansive self mapping $T : E \longrightarrow K(E)$, where E is nonempty closed convex subset of a Banach X space which has the property that asymptotic center in E of each bounded sequence of X is nonempty and compact. ln 2001, Xu [29] extended Kirk and Massa's work to nonself mapping $T : E \longrightarrow K(X)$ which satisfies an inwardness condition.

Let $T : C \longrightarrow P(C)$ be multivalued mapping. Define

$$P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}, \forall x \in K. \tag{1.14}$$

Then, P_T is called best approximation operator.

Approximating fixed points of multivalued mappings using best approximation method was first introduced by Hussan and Khan [33] in 2003.They proved that if C is a closed unbounded subset of a Hilbert space H, then every $\star$-nonexpansive multivalued mapping has a fixed point. Since then, numerous results have been proved for multivalued mappings using best approximation operator in Banach spaces (see, for example, [26], [30], [31], [35], [42] and the references therein).

For sometime now, it has been observed that approximation of fixed points of multivalued mappings $T : D(T) \subseteq E \longrightarrow 2^E$ with regard to Hausdorff metric have been impossible without imposing the conditions that either the fixed point set of T is strict or T is a multivalued mapping for which P_T satisfies some contractive conditions. Consequently, it becomes natural to ask if there exists a class of multivalued mappings with nonempty fixed point set for which neither the fixed point set of T is strict nor P_T satisfies any contractive condition so far studied by some authors?

Isiogugu [40] gave an affirmative answer to the above question and also noted that this class of mapping possesses other interesting properties of some existing maps recently studied in ([26],[30], [31], [35], [42] and the references therein). In line with this observation, a new type of map, which she called type-one mapping (Recall that a multivalued map $S : D(S) \subseteq X \longrightarrow 2^X$ defined on a normed space E is called a type one if given any pair $r, g \in D(S)$, we have $\|v - v\| \leq \Phi(Sr, Sg), \forall u \in P_S r, v \in P_S g)$, was introduced and used in direct approximation of fixed points for multivalued mappings.

In all the papers considered above, the iteration scheme used involved multivalued operators only. In 2011, Sokhuma and Kaevkhao [37] was the first who introduced the iteration scheme (1.15) for approximating a common fixed point of a pair of nonexpansive single valued mapping t and nonexpansive multivalued mapping T:

$$\left.\begin{aligned} &x_0 \in E\\ &y_n = (1-\alpha_n)x_n + \alpha_n z_n,\\ &x_{n+1} = (1-\beta_n)x_n + \beta_n t y_n, n \in N, \end{aligned}\right\} \tag{1.15}$$

where $z_n \in Tx_n$. They pointed out that the control condition $0 < a \leq \alpha_n, \beta_n \leq b < 1$ is necessary for convergence of the iteration scheme (1.15) to the common fixed point $p \in F(t) \cap F(T)$ in the setting of uniformly convex Banach spaces. To be precise, they proved the following theorems:

Theorem 1.9. *(SK_1) Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and let $t : E \longrightarrow E$ and $T : E \longrightarrow FB(E)$ be single-valued and multivalued nonexpansive mappings, respectively, and satisfying $Tw = \{w\}$ for all $w \in F(t) \cap F(T)$. Let x_n be the sequence of the modified Ishikawa iteration defined by*

$$\left.\begin{aligned} &x_0 \in E\\ &y_n = (1-\alpha_n)x_n + \alpha_n z_n,\\ &x_{n+1} = (1-\beta_n)x_n + \beta_n t y_n, n \in N, \end{aligned}\right\}$$

where $z_n \in Tx_n$. If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in F(t) \cap F(T)$.

Theorem 1.10. *(SK_2) Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and let $t : E \longrightarrow E$ and $T : E \longrightarrow FB(E)$ be single-valued and multivalued nonexpansive mappings, respectively, and satisfying $Tw = \{w\}$ for all $w \in F(t) \cap F(T)$. Let x_n be the sequence of the modified Ishikawa iteration defined by*

$$\left.\begin{aligned} &x_0 \in E\\ &y_n = (1-\alpha_n)x_n + \alpha_n z_n,\\ &x_{n+1} = (1-\beta_n)x_n + \beta_n t y_n, n \in N, \end{aligned}\right\}$$

where $z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of T.

In [38], using iteration scheme for a pair of a finite family of asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi nonexpansive multivalued mapping $\{T_i\}_{i=1}^N$, Eslamia [38] extended the results of [37] in uniformly convex Banach spaces with the following iteration scheme:

$$\left.\begin{aligned} &x_1 \in D\\ &y_n = \beta_n^0)x_n + \textstyle\sum_{i=1}^N \beta_n^{(i)} z_n^{(i)},\\ &x_{n+1} = \alpha_n^0)x_n + \textstyle\sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, n \in N, \end{aligned}\right\} \tag{1.16}$$

where $z_n^{(i)} \in T_i x_n$ and $\alpha_n^{(i)}, \beta_n^{(i)\in[0,1]}$ such that $\sum_{i=1}^N \beta_n^{(i)} = 1 = \sum_{i=1}^N \alpha_n^{(i)}$, for $i = 1, 2, \cdots, N$. Rencently, Suantal and Phuengrattana [36] extended the results of [38] in uniformly convex Banach space using (1.16) but for a pair of a finite family of generalised asymptotically nonexpansive single-valued mappings and a finite family of quasi nonexpansive multivalued mappings. More precisely, they proved the following theorems:

Theorem 1.11. (SP_1) *Let D be a nonempty,compact and convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitizian and generalised asymptotically nonexpansive single-valued mappings of D into itself with sequences $k_n \subset [1,\infty)$ and $s_n \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multivalued mappings of D into $CB(D)$ satisfying condition E. Assume that $\cap_{i=1}^N F(t_i) \cap \cap_{i=1}^N F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in F$ and $i = 1,2,\cdots,N$. Let $\{x_n\}$ be a sequence generated by*

$$\left.\begin{aligned} &x_1 \in D \\ &y_n = \beta_n^0)x_n + \textstyle\sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ &x_{n+1} = \alpha_n^0)x_n + \textstyle\sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, n \in N, \end{aligned}\right\}$$

where $z_n^{(i)} \in T_i x_n$ and $\alpha_n^{(i)}, \beta_n^{(i)\in[0,1]}$ such that $\sum_{i=1}^N \beta_n^{(i)} = 1 = \sum_{i=1}^N \alpha_n^{(i)}$, for $i = 1,2,\cdots,N$. Then $\{x_n\}$ converges strongly to a fixed point in F.

Theorem 1.12. (SP_2) *Let D be a nonempty,close and convex subset of a uniformly convex Banach space X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitizian and generalised asymptotically nonexpansive single-valued mappings of D into itself with sequences $k_n \subset [1,\infty)$ and $s_n \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multivalued mappings of D into $CB(D)$ satisfying condition E. Assume that $\cap_{i=1}^N F(t_i) \cap \cap_{i=1}^N F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in F$ and $i = 1,2,\cdots,N$. Let $\{x_n\}$ be a sequence generated by*

$$\left.\begin{aligned} &x_1 \in D \\ &y_n = \beta_n^0)x_n + \textstyle\sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ &x_{n+1} = \alpha_n^0)x_n + \textstyle\sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, n \in N, \end{aligned}\right\}$$

where $z_n^{(i)} \in T_i x_n$ and $\alpha_n^{(i)}, \beta_n^{(i)\in[0,1]}$ such that $\sum_{i=1}^N \beta_n^{(i)} = 1 = \sum_{i=1}^N \alpha_n^{(i)}$, for $i = 1,2,\cdots,N$. Then $\{x_n\}$ converges weakly to a fixed point in F.

The class of total asymptotically quasi-nonexpansive single-valued and multivalued nonself mappings is a supperclass of the following classes of mappings: total asymptotically quasi-nonexpansive single-valued and multivalued self mappings, generalised asymptotically quasi-nonexpansive single-valued and multivalued self (and nonself) mappings, asymptotically quasi-nonexpansive single-valued and multivalued self (and nonself) mappings, quasi-nonexpansive single-valued and multivalued self (and nonself) mappings, nonexpansive single-valued and multivalued self (and nonself) mappings, and a host of other self (and nonself) mappings. Thus, the study of the fixed point of total asymptotically quasi-nonexpansive single-valued and multivalued nonself mappings may help in the study of the fixed point of the above mentioned mappings. Part of the novelty of the paper is that the results obtained using our iteration scheme generalize all other results obtained so far in literature for every mappings connected to the class of asymptotically mutivalued nonexpansive mappings in uniformly convex Banach spaces.

Again, the mappings T_i and t_i, for $i = 1, 2, \cdots, N$, used in the recursion formula of (1.15) and (1.16) are both self mappings. If, however, the domain $D(T_i)$ and $D(t_i)$ of the operators T_i and t_i are proper subsets of the Banach space E and (T_i, t_i) maps their respective domains into E, then the two recursion formula may fail to be well defined. Thus, the following question arises:

Question 1.1. Can iterative algorithm for which the mapping of a proper subset of a Banach space E into E that will generalize (1.15) and (1.16) (and still remain well defined) be constructed such that a much more general results could be obtained?

It is our purpose in this paper to first introduce the important class of total asymptotically quasi-nonexpansive multivalued nonself mappings which is much more general than all the classes of mappings studied in this paper. Then, we prove strong and weak convergence theorems for this class of mappings in the setting of uniformly convex Banach space. The iterative scheme employed to make this happen is the following:

Let E be a real Banach space, D a nonempty closed convex subset of E and $R : E \longrightarrow K$ a nonexpansive retraction of E onto K. Let $\{G_i\}_{i=1}^N : D \longrightarrow E$, $i = 1, 2, \cdots$, N, be a finite family of total asymptotically nonexpansive single-valued nonself mappings and $\{T_i\}_{i=1}^N : D \longrightarrow 2^E$, for $i = 1, 2, \cdots$, N, be a finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings. Then, the hybrid iteration scheme for the above mentioned mappings is as follows:

$$\left.\begin{aligned} x_1 &= x \in D; \\ x_{n+1} &= R(\alpha_{n,0}x_n + \textstyle\sum_{i=1}^N \alpha_{n,i}v_{n,i}); \\ y_n &= R(\beta_{n,0}x_n + \textstyle\sum_{i=1}^N \beta_{n,i}G_i(RG_i)^{n-1}y_n, \forall n \in N, \end{aligned}\right\} \tag{1.17}$$

where $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ are real sequences in $[0, 1]$, for $i = 1, 2, \cdots$, N, $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1}y_n$ *with* $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1}y_n)$.

2 Preliminary

For the sake of convenience, we restate the following concepts and results:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0, 2] \longrightarrow (0, 2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x - y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$.

We recall the following:

Definition 2.1. *A multivalued mapping $T : D \longrightarrow CB(D)$ is said to satisfy condition E_μ, where $\mu \geq 0$, if for each $x, y \in D$,*

$$d(x, Ty) \leq \mu d(x, Tx) + \|x - y\|.$$

We say that T satisfy condition (E) whenever T satisfies (E_μ) for some $\mu \geq 0$.

Definition 2.2. *The space E has Opial condition* [10] *if for any sequence $\{x_n\}$ in E, x_n converges to x weakly,it follows that* $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ *for all $y \in E$ with $x \neq y$.*

Examples of Banach spaces satisfying Opial conditions are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, \pi]$ with $1 < p \neq 2$ fails to satify Opial condition.

Definition 2.3. [34] *A multivalued mapping $T : D \longrightarrow \mathcal{P}(D)$ is said to satisfy condition (1) if the exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f > 0$on $(0, \infty)$ such that*

$$d(x, Tx) \geq f(d(x, F(T)), \forall x \in D. \tag{2.1}$$

Definition 2.4. *(see [36]) Let F be a nonempty suset of a Banach space X and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is of monotone type (I) with respect to F if there exist sequences $\{\delta_n\}$ and $\{xi_n\}$ of nonnegative real numbers suc that $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$ and $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + \xi_n$ for all $n \in N$ and $p \in F$.*

Next, we state the following useful lemmas and propositions to prove our main results.

Lemma 2.5. $(see[16])$*: Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality:*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + \gamma_n, \forall n \geq 1. \tag{2.2}$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then

1. $\lim_{n\to\infty} \alpha_n$ *exists*
2. *In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to 0, then* $\lim_{n\to\infty} = 0$.

Lemma 2.6. $(see[14])$*: Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for each $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r \text{ and } \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r, \tag{2.3}$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.7. *(see [36]) Let X be the Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\|x_n - u\|$ and $\|x_n - v\|$ exists. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converges to u and v respectively, then $u = v$.*

Proposition 2.8. *(see [36]) Let X be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing continous function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| - \lambda(1-\lambda)g(\|x-y\|)$ for all $x, y \in B_r = \{z \in X : \|z\| \leq r, \lambda \in [0,1]\}$.*

Proposition 2.9. *(see [36]) Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is of monotone type (I) with respect to F and $d(x_n, F) = 0$, then $\lim_{n\to\infty} x_n = p$ for some $p \in X$ satisfying $d(x_n - F) = 0$. In particular, if F is closed, then $p \in F$.*

3 Main Results

In the sequel, we need the following definitions and lemmas to prove our main results.

Definition 3.1. *Let D be a nonempty closed convex subset of a Banach space E. A nonself multivalued mapping $T : D \to CB(E)$ is said to be total asymptotically nonexpansive, in the sense Yolacan and Kiziltune [18],if there exist sequences $k_n^{(1)}$ and $k_n^{(2)} \subset [0, \infty) : k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that*

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(y)) \leq \|x-y\| + k_n^{(1)}\phi(\|x-y\|) + k_n^{(2)}, \tag{3.1}$$

for all $x, y \in K$ and $\forall n \in \mathbb{N}$.

Remark 3.2. *If $\phi(t) = t$, then (3.1) becomes*

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(y)) \leq (1 + k_n^{(1)})\|x-y\| + k_n^{(2)}, \forall n \geq 1, \tag{3.2}$$

and is called generalize asymptotically nonexpansive multivalued nonself mapping. Again, if $\phi(t) = t$ and $k_n^{(2)} = 0$, then (3.1) reduces to

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(y)) \leq \alpha_n\|x-y\|, \forall n \geq 1, \tag{3.3}$$

and is called asymptotically nonexpansive multivalued nonself mapping with $\alpha_n = 1 + k_n^{(1)} \to 1$ as $n \to \infty$.

Note that if the fixed point set of a total asymptotically nonexpansive multivalued nonself mapping is nonempty, then we have the following definition:

Definition 3.3. *Let D be a nonempty closed convex subset of a Banach space E. A nonself multivalued mapping $T : K \to CB(E)$ is said to be total asymptotically quasi-nonexpansive , in the sense Yolacan and Kiziltune [18],if there exist sequences $k_n^{(1)}$ and $k_n^{(2)} \subset [0, \infty) : k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $q \in F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$, we have*

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(q)) \leq \|x-q\| + k_n^{(1)}\phi(\|x-q\|) + k_n^{(2)}, \tag{3.4}$$

$\forall x \in K$ and $n \in \mathbb{N}$.

Remark 3.4. *If $\phi(t) = t$, then (3.4) becomes*

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(q)) \leq (1 + k_n^{(1)})\|x-q\| + k_n^{(2)}, \forall n \geq 1, \tag{3.5}$$

and is called generalize asymptotically quasi-nonexpansive multivalued nonself mapping. Again, if $\phi(t) = t$ *and* $k_n^{(2)} = 0$*, then (3.4) reduces to*

$$H(T(RT)^{n-1}(x), T(RT)^{n-1}(q)) \leq \alpha_n \|x - q\|, \forall n \geq 1, \tag{3.6}$$

and is called asymptotically quasi-nonexpansive multivalued nonself mapping with $\alpha_n = 1 + k_n^{(1)} \to 1$ *as* $n \to \infty$.

Lemma 3.5. *Let E be a Banach space and D a nonempty, closed and convex subset of E. Let* $\{G_i\}_{i=1}^N : D \longrightarrow E$ *be a finite family of total asymptotically nonexpansive single-valued mappings with sequences* $\{\nu_{n,i}\}_{n\geq 1}, \{\omega_{n,i}\}_{n\geq 1} \to 0$ *as* $n \to \infty$, $for\ i = 1, 2, \cdots, \text{N}$. *Let* $\{T_i\}_{i=1}^N : D \longrightarrow CB(E)$ *be a type one finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings from D into the family of close and bounded subsets of E with sequences* $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ *as* $n \to \infty$, $for\ i = 1, 2, \cdots, \text{N}$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\left.\begin{aligned} &x_1 = x \in D; \\ &\quad x_{n+1} = R(\alpha_{n,0} x_n + \textstyle\sum_{i=1}^N \alpha_{n,i} v_{n,i}); \\ &\quad y_n = R(\beta_{n,0} x_n + \textstyle\sum_{i=1}^N \beta_{n,i} G_i (RG_i)^{n-1} y_n, \forall n \in N \end{aligned}\right\} \tag{3.7}$$

where $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ *are real sequences in* $[0, 1]$, *for* $i = 1, 2, \cdots, \text{N}$, $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1} y_n$ *with* $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1} y_n)$. *Assume* $F = \cap_{i=1}^N (F(G_i) \cap F(T_i)) \neq \emptyset$ *and closed. If the following conditions hold:*

i. $\sum_{n=1}^\infty \mu_{n,i} < \infty, \sum_{n=1}^\infty \xi_{n,i} < \infty, \sum_{n=1}^\infty \nu_{n,i} < \infty, \sum_{n=1}^\infty \omega_{n,i} < \infty;$

ii. *There exist nonnegative strictly increasing continuous functions* $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ *with* $\phi(0) = 0 = \psi(0)$ *and some constants* $M_i', M_i'' > 0$ *such that* $\psi(t) \leq M_i' t$ *and* $\phi(s) = M'' s$, $for\ i = 1, 2, \cdots N, \forall t, s \geq 0$.

Then, $\lim_{n\infty} \|x_n - q\|$ *exists for all* $q \in F$.

Proof. Set $\tau_n = \max_{1\leq i\leq N}\{\mu_{n,i}, \nu_{n,i}\}, M = \max_{1\leq i\leq N}\{M_i', M_i''\}$ and $\theta_n = \max_{1\leq i\leq N}\{\xi_{n,i}, \omega_{n,i}\}$. Then,

$\sum_{n=1}^{\infty} \tau_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. For any $q \in F$, it follows from (3.7) that

$$
\begin{aligned}
\|y_n - q\| &= \|R(\beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}v_{n,i}) - q\| \\
&\leq \|\beta_{n,0}(x_n - q) + \sum_{i=1}^{N} \beta_{n,i}(v_{n,i} - q) + \sum_{i=1}^{N} \beta_{n,i}q - q\| \\
&= \|\beta_{n,0}(x_n - q) + \sum_{i=1}^{N} \beta_{n,i}(v_{n,i} - q)\| \\
&\leq \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\|v_{n,i} - q\| \\
&\leq \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}H(T_i(RT_i)^{n-1}x_n, T_i(RT_i)^{n-1}q) \\
&\leq \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}[\|x_n - q\| + \mu_{n,i}\psi(\|x_n - q\|) + \xi_{n,i}] \\
&= \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\mu_{n,i}\psi(\|x_n - q\|) + \sum_{i=1}^{N} \beta_{n,i}\xi_{n,i} \\
&\leq \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\mu_{n,i}M_i'\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\xi_{n,i} \\
&\leq \beta_{n,0}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\tau_n M\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\theta_n \\
&= \|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\tau_n M\|x_n - q\| + \sum_{i=1}^{N} \beta_{n,i}\theta_n \\
&= \|x_n - q\| + (\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i})\tau_n M\|x_n - q\| + (\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i})\theta_n \\
&\quad -\beta_{n,0}(\tau_n M\|x_n - q\| + \theta_n) \\
&\leq (1 + \tau_n M)\|x_n - q\| + \theta_n \qquad (3.8)
\end{aligned}
$$

Also, from (3.7), we obtain

$$\begin{aligned}
\|x_{n+1}-q\| &= \|R(\alpha_{n,0}x_n+\sum_{i=1}^{N}\alpha_{n,i}G_i(RG_i)^{n-1}y_n)-q\| \\
&\leq \|\alpha_{n,0}(x_n-q)+\sum_{i=1}^{N}\alpha_{n,i}(G_i(RG_i)^{n-1}y_n-q)+\sum_{i=1}^{N}\alpha_{n,i}q-q\| \\
&= \|\alpha_{n,0}(x_n-q)+\sum_{i=1}^{N}\alpha_{n,i}(G_i(RG_i)^{n-1}y_n-q)\| \\
&\leq \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\|G_i(RG_i)^{n-1}y_n-q\| \\
&\leq \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}[\|y_n-q\|+\nu_{n,i}\phi(\|y_n-q\|)+\omega_{n,i}] \\
&= \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\nu_{n,i}\phi(\|x_n-q\|)+\sum_{i=1}^{N}\alpha_{n,i}\omega_{n,i} \\
&\leq \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\nu_{n,i}M_i''\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\omega_{n,i} \\
&\leq \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\tau_nM\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\theta_n \\
&= \alpha_{n,0}\|x_n-q\|+(1+\tau_nM)\sum_{i=1}^{N}\alpha_{n,i}\|y_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\theta_n \qquad (3.9)
\end{aligned}$$

(3.8) and (3.9) imply that

$$\begin{aligned}
\|x_{n+1}-q\| &\leq \alpha_{n,0}\|x_n-q\|+(1+\tau_nM)\sum_{i=1}^{N}\alpha_{n,i}[(1+\tau_nM)\|x_n-q\| \\
&\quad +\theta_n]+\sum_{i=1}^{N}\alpha_{n,i}\theta_n \\
&= \alpha_{n,0}\|x_n-q\|+(1+\tau_nM)^2\sum_{i=1}^{N}\alpha_{n,i}\|x_n-q\|+\theta_n(1+\tau_nM)\sum_{i=1}^{N}\alpha_{n,i} \\
&\quad +\sum_{i=1}^{N}\alpha_{n,i}\theta_n \\
&= \alpha_{n,0}\|x_n-q\|+\sum_{i=1}^{N}\alpha_{n,i}\|x_n-q\|+(2+\tau_nM)\tau_nM(\alpha_{n,0}+\sum_{i=1}^{N}\alpha_{n,i})\|x_n-q\| \\
&\quad +(\alpha_{n,0}+\sum_{i=1}^{N}\alpha_{n,i})(2+\tau_nM)\theta_n-\alpha_{n,0}(2+\tau_nM)(\theta_n+\tau_nM\|x_n-q\|) \\
&\leq [1+(2+\tau_nM)\tau_nM]\|x_n-q\|+\theta_n(1+\tau_nM) \\
&= (1+\delta_n)\|x_n-q\|+\rho_n \qquad (3.10)
\end{aligned}$$

where $\delta_n=(2+\tau_nM)\tau_nM$ and $\rho_n=\theta_n(1+\tau_nM)$.

Since $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\rho_n<\infty$, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists. This completes the proof. □

Lemma 3.6. *Let E be a Banach space and D a nonempty, closed and convex subset of E. Let $\{G_i\}_{i=1}^{N}:D\longrightarrow E$ be an L-Lipschitizian and a finite family of total asymptotically nonexpansive single-valued mappings with sequences*

$\{\nu_{n,i}\}_{n\geq 1}, \{\omega_{n,i}\}_{n\geq 1} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots$, N. Let $\{T_i\}_{i=1}^N : D \longrightarrow CB(E)$ be a type one finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings from D into the family of close and bounded subsets of E with sequences $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots$, N. Let $\{x_n\}$ be the sequence defined by

$$\left.\begin{aligned} &x_1 = x \in D; \\ &\qquad x_{n+1} = R(\alpha_{n,0}x_n + \textstyle\sum_{i=1}^N \alpha_{n,i}v_{n,i}); \\ &\qquad y_n = R(\beta_{n,0}x_n + \textstyle\sum_{i=1}^N \beta_{n,i}G_i(RG_i)^{n-1}y_n, \forall n \in N \end{aligned}\right\} \tag{3.11}$$

where $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are real sequences in $[0, 1]$, for $i = 1, 2, \cdots$, N, $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1}y_n$ with $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1}y_n)$. Assume $F = \cap_{i=1}^N (F(G_i) \cap F(T_i)) \neq \emptyset$ and closed. If the following conditions hold:

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^\infty \mu_{n,i} < \infty, \sum_{n=1}^\infty \xi_{n,i} < \infty, \sum_{n=1}^\infty \nu_{n,i} < \infty, \sum_{n=1}^\infty \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0 = \psi(0)$ and some constants $M_i', M_i'' > 0$ such that $\psi(t) \leq M_i't$ and $\phi(s) = M''s$, for $i = 1, 2, \cdots N, \forall t, s \geq 0$.*

Then, $\lim_{n\to\infty} \|x_n - v_{n,i}\| = 0 = \lim_{n\to\infty} \|x_n - G_ix_n\| = 0$, for $i = 1, 2, , \cdots, N$.

Proof. Set $\tau_n = \max_{1\leq i\leq N}\{\mu_{n,i}, \nu_{n,i}\}, M = \max_{1\leq i\leq N}\{M_i', M_i''\}$ and $\theta_n = \max_{1\leq i\leq N}\{\xi_{n,i}, \omega_{n,i}\}$. Then, $\sum_{n=1}^\infty \tau_n < \infty$ and $\sum_{n=1}^\infty \theta_n < \infty$. For $q \in F$, $\lim_{n\to\infty} \|x_n - q\|$ exists by Lemma 3.1. Now, assume that $\lim_{n\to\infty} \|x_n - q\| = c \geq 0$. If c=0, then there is nothing left to prove. Hence, let $c > 0$ so that we have the following estimates: By definition, we have

$$\begin{aligned} \|G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}q\| &\leq \|y_n - q\| + \mu_{n,i}\phi(\|y_n - q\|) + \xi_{n,i} \\ &\leq \|y_n - q\| + \tau_nM\|y_n - q\|) + \theta_n \\ &= (1 + \tau_nM)\|y_n - q\| + \theta_n \end{aligned} \tag{3.12}$$

From (3.8) and (3.12), it follows that

$$\begin{aligned} \|G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}q\| &\leq (1 + \tau_nM)[(1 + \tau_nM)\|x_n - q\| + \theta_n] + \theta_n \\ &= (1 + \tau_nM)^2\|x_n - q\| + (2 + \tau_nM)\theta_n \end{aligned} \tag{3.13}$$

Since $\sum_{n=1}^\infty \tau_n < \infty$ *and* $\sum_{n=1}^\infty \theta_n < \infty$, we get

$$\limsup_{n\to\infty} \|G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}q\| \leq c, \text{ for } i = 1, 2, \cdots, N. \tag{3.14}$$

Furthermore, from the fact $\lim_{n\to\infty} \|x_n - q\| = c$ and that

$$c = \lim_{n\to\infty} \|x_{n+1} - q\| = \|\alpha_{n,i}(x_n - q) + \sum_{i=1}^N \alpha_{n,i}(G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}q)\|, \tag{3.15}$$

it follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - G_i(RG_i)^{n-1}y_n\| = 0,\ for\ i = 1, 2, \cdots, N. \tag{3.16}$$

Moreover, from (3.9), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \alpha_{n,0}\|x_n - q\| + (1+\tau_n M)\sum_{i=1}^{N}\alpha_{n,i}\|y_n - q\| + \sum_{i=1}^{N}\alpha_{n,i}\theta_n \\
&= (\alpha_{n,0} + \sum_{n=1}^{N}\alpha_{n,i} - \sum_{n=1}^{N}\alpha_{n,i})\|x_n - q\| + (1+\tau_n M)\sum_{i=1}^{N}\alpha_{n,i}\|y_n - q\| + \sum_{i=1}^{N}\alpha_{n,i}\theta_n \\
&= (1 - \sum_{n=1}^{N}\alpha_{n,i})\|x_n - q\| + (1+\tau_n M)\sum_{i=1}^{N}\alpha_{n,i}\|y_n - q\| + \sum_{i=1}^{N}\alpha_{n,i}\theta_n
\end{aligned}$$

The last inequality implies that

$$\|x_{n+1} - q\| - \|x_n - q\| \leq \sum_{i=1}^{N}\alpha_{n,i}[(1+\tau_n M)\|y_n - q\| + \theta_n - \|x_n - q\|]$$

Consequently,

$$\begin{aligned}
\frac{\|x_{n+1} - q\| - \|x_n - q\|}{bN} + \|x_n - q\| &\leq \frac{\|x_{n+1} - q\| - \|x_n - q\|}{\sum_{i=1}^{N}\alpha_{n,i}} + \|x_n - q\| \\
&\leq (1+\tau_n M)\|y_n - q\| + \theta_n
\end{aligned}$$

Thus,

$$\begin{aligned}
c &= \limsup_{n\to\infty} \frac{\|x_{n+1} - q\| - \|x_n - q\|}{bN} + \|x_n - q\| \\
&\leq \liminf_{n\to\infty}(1+\tau_n M)\|y_n - q\| + \theta_n \\
&= \liminf_{n\to\infty}\|y_n - q\| \\
&\leq c \qquad (by (3.8))
\end{aligned} \tag{3.17}$$

(3.17) implies that

$$c = \|y_n - q\| = \|\beta_{n,0}(x_n - q) + \sum_{i=1}^{N}\beta_{n,i}(u_{n,i} - q)\| \tag{3.18}$$

Since,

$$\begin{aligned}
\|v_{n,i} - q\| &\leq H(T_i(RT_i)^{n-1}x_n - q) \\
&\leq \|x_n - q\| + \nu_{n,i}\phi(\|x_n - q\|) + \omega_{n,i} \\
&\leq \|x_n - q\| + \tau_n M\|x_n - q\| + \theta_n \\
&\leq (1+\tau_n M)\|x_n - q\| + \theta_n,
\end{aligned}$$

it follows that

$$\limsup_{n\to\infty}\|v_{n,i} - q\| \leq c \tag{3.19}$$

Thus, from (3.18),(3.19), Lemma 2.2 and the fact that $\lim_{n\to\infty}\|x_n - q\| = c$, we get

$$\lim_{n\to\infty}\|x_n - v_{n,i}\| = 0,\ for\ i = 1, 2, \cdots, N. \tag{3.20}$$

Also, since $v_{n,i} \in T_i(RT_i)^{n-1}x_n$, it follows that

$$\lim_{n\to\infty} d(x_n, T_i(RT_i)^{n-1}x_n) = 0, \; for \; i = 1, 2, \cdots, N. \tag{3.21}$$

In addition, from (3.7), we have

$$\|y_n - x_n\| = \sum_{i=1}^{N} \beta_{n,i}\|u_{n,i} - x_n\|, \; for \; i = 1, 2, \cdots, N, \tag{3.22}$$

which by (3.20) gives

$$\lim_{n\to\infty} \|y_n - x_n\| = 0 \tag{3.23}$$

Also, observe that

$$\begin{aligned} \|x_n - G_i(RG_i)^{n-1}x_n\| &\leq \|x_n - G_i(RG_i)^{n-1}y_n\| + \|G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}x_n\| \\ &\leq \|x_n - x_n - G_i(RG_i)^{n-1}y_n\| + L\|y_n - x_n\| \end{aligned}$$

The last inequality implies that, for $i = 1, 2, \cdots, N$,

$$\lim_{n\to\infty} \|x_n - G_i(RG_i)^{n-1}x_n\| = 0, \qquad (by(3.16)and(3.23)). \tag{3.24}$$

Again, from (3.7), we have the following estimares:

$$\|x_{n+1} - x_n\| = \sum_{i=1}^{N} \alpha_{n,i}\|v_{n,i} - x_n\| \Rightarrow \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \qquad (by(3.28)); \tag{3.25}$$

From (3.23), (3.25) and the inequality below

$$\|y_n - x_{n-1}\| \leq \|y_n - x_n\| + \|x_n - x_{n-1}\|,$$

we get

$$\lim_{n\to\infty} \|y_n - x_{n-1}\| = 0; \tag{3.26}$$

and from (3.23), (3.25) and the inequality below

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - y_n\|,$$

we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.27}$$

Now, we show that $\|x_n - G_ix_n\| = 0$: To do this, observe that

$$\begin{aligned} \|x_n - G_ix_n\| &\leq \|x_n - G_i(RG_i)^{n-1}y_n\| + \|G_i(RG_i)^{n-1}y_n - G_i(RG_i)^{n-1}x_{n-1}\| \\ &\quad + \|G_i(RG_i)^{n-1}x_{n-1} - G_ix_n\| \\ &\leq \|x_n - G_i(RG_i)^{n-1}y_n\| + L\|y_n - x_{n-1}\| + L\|(RG_i)(RG_i)^{n-2}x_{n-1} - x_n\| \\ &\leq \|x_n - G_i(RG_i)^{n-1}y_n\| + L\|y_n - x_{n-1}\| + L\|G_i(RG_i)^{n-2}x_{n-1} - x_{n-1}\| \\ &\quad + L\|x_{n-1} - x_n\| \end{aligned} \tag{3.28}$$

From (3.16), (3.21),(3.25) and (3.26), we have

$$\lim_{n\to\infty} \|x_n - G_ix_n\| = 0, \qquad \text{for } i = 1, 2, \cdots, N. \tag{3.29}$$

This completes the proof. □

Theorem 3.7. *Let E be a uniformly convex Banach space and D a nonempty, compact and convex subset of E. Let $\{G_i\}_{i=1}^N : D \longrightarrow E$ be an L-Lipschitizian and finite family of total asymptotically nonexpansive single-valued nonself mappings with sequences $\{\nu_{n,i}\}_{n\geq 1}, \{\omega_{n,i}\}_{n\geq 1} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots, N$. Let $\{T_i\}_{i=1}^N : D \longrightarrow CB(E)$ be an L-Lipschitizian and type one finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings from D into the family of closed and bounded subsets of E with sequences $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots, N$. Let $\{x_n\}$ be the sequence defined by*

$$\left.\begin{array}{l} x_1 = x \in D; \\ \qquad x_{n+1} = R(\alpha_{n,0}x_n + \sum_{i=1}^N \alpha_{n,i}v_{n,i}); \\ \qquad y_n = R(\beta_{n,0}x_n + \sum_{i=1}^N \beta_{n,i}G_i(RG_i)^{n-1}y_n, \forall n \in N \end{array}\right\} \tag{3.30}$$

where $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are real sequences in $[0, 1]$, for $i = 1, 2, \cdots, N$, $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1}y_n$ with $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1}y_n)$. Suppose $\{T_i\}_{i=1}^n$ satisfies Condition (E) and $F = \cap_{i=1}^N (F(G_i) \cap F(T_i)) \neq \emptyset$. If the following conditions hold:

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^\infty \mu_{n,i} < \infty, \sum_{n=1}^\infty \xi_{n,i} < \infty, \sum_{n=1}^\infty \nu_{n,i} < \infty, \sum_{n=1}^\infty \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0 = \psi(0)$ and some constants $M_i', M_i'' > 0$ such that $\psi(t) \leq M_i't$ and $\phi(s) = M''s$, for $i = 1, 2, \cdots N, \forall t, s \geq 0$.*

Then, the $\{x_n\}_{n\geq 1}$ converges strongly to a point of F

Proof. From Lemma 3.1, the sequence $\{x_n\}_{n\geq 1}$ is bounded. Since D is compact, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ which converges to $q \in D$. By Condition (E), there exists $\mu \geq 1$ such that, for $i = 1, 2, \cdots, N$,

$$\begin{aligned} d(q, T_i(RT_i)^{n-1}q) &\leq \|q - x_{n_k}\| + d(x_{n_k}, T_i(RT_i)^{n-1}x_{n_k}) \\ &\leq \|q - x_{n_k}\| + \mu d(x_{n_k}, T_i(RT_i)^{n-1}x_{n_k}) + 2\|x_{n_k} - q\| \\ &= 2\|x_{n_k} - q\| + \mu d(x_{n_k}, T_i(RT_i)^{n-1}x_{n_k}) \end{aligned}$$

The last inequality implies that

$$\lim_{n\to\infty} d(q, T_i(RT_i)^{n-1}q) = 0 \tag{3.31}$$

Next, from the inequality

$$\begin{aligned} d(q, T_iq) &\leq d(q, T_i(RT_i)^{n-1}q) + d(T_i(RT_i)^{n-1}q, T_i(RT_i)^{n-1}x_{n_k-1}) \\ &\quad + d(T_i(RT_i)^{n-1}x_{n_k-1} - T_iq) \\ &\leq d(q, T_i(RT_i)^{n-1}q) + L\|q - x_{n_k-1}\| + d(T_i(RT_i)(RT_i)^{n-2}x_{n_k-1} - T_iq) \\ &\leq d(q, T_i(RT_i)^{n-1}q) + L\|q - x_{n_k-1}\| + Ld((RT_i)(RT_i)^{n-2}x_{n_k-1} - q) \\ &\leq d(q, T_i(RT_i)^{n-1}q) + 2L\|q - x_{n_k-1}\| + Ld(T_i(RT_i)^{n-2}x_{n_k-1} - x_{n_k-1}), \end{aligned}$$

(3.21) and (3.31), we get

$$\lim_{n\to\infty} d(q, T_i q) = 0 \tag{3.32}$$

$\Rightarrow q \in \cap_{i=1}^{N} F(T_i)$. Therefore, $q \in F$.
This completes the proof. $\square$

Theorem 3.8. *Let D a nonempty, closed and convex subset of a real uniformly convex Banach space E with the Opial property. Let $\{G_i\}_{i=1}^{N} : D \longrightarrow E$ be an L-Lipschitizian and finite family of total asymptotically nonexpansive single-valued nonself mappings with sequences $\{\nu_{n,i}\}_{n\geq1}, \{\omega_{n,i}\}_{n\geq1} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots, N$. Let $\{T_i\}_{i=1}^{N} : D \longrightarrow KC(E)$ be an L-Lipschitizian and a type one finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings from D into the family of compact convex subsets of E with sequences $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ as $n \to \infty$, for $i = 1, 2, \cdots, N$. Let $\{x_n\}$ be the sequence defined by*

$$\left.\begin{array}{l} x_1 = x \in D; \\ \qquad x_{n+1} = R(\alpha_{n,0}x_n + \sum_{i=1}^{N} \alpha_{n,i}v_{n,i}); \\ \qquad y_n = R(\beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}G_i(RG_i)^{n-1}y_n, \forall n \in N \end{array}\right\} \tag{3.33}$$

where $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are real sequences in $[0,1]$, for $i = 1, 2, \cdots, N$, $\sum_{i=0}^{N} \alpha_{n,i} = 1 = \sum_{i=0}^{N} \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1}y_n$ with $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1}y_n)$. Suppose $\{T_i\}_{i=1}^{n}$ satisfies Condition (E) and $F = \cap_{i=1}^{N}(F(G_i) \cap F(T_i)) \neq \emptyset$. If the following conditions hold:

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^{\infty} \mu_{n,i} < \infty, \sum_{n=1}^{\infty} \xi_{n,i} < \infty, \sum_{n=1}^{\infty} \nu_{n,i} < \infty, \sum_{n=1}^{\infty} \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0 = \psi(0)$ and some constants $M_i', M_i'' > 0$ such that $\psi(t) \leq M_i' t$ and $\phi(s) = M'' s$, for $i = 1, 2, \cdots N, \forall t, s \geq 0$.*

Then, the $\{x_n\}_{n\geq1}$ converges weakly to a point of F

Proof. From Lemma 3.1, $\{x_n\}_{n\geq1}$ is bounded. Since E is a uniformly convex Banach space, there exists a subsequence $\{x_{n_k}\}_{k\geq1}$ of $\{x_n\}_{n\geq1}$ converging to $q \in D$. From Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - v_{n,i}\| = 0$ and $\lim_{n\to\infty} \|x_n - G_i x_n\| = 0$, for $i = 1, 2, \cdots, N$. We need to show that $q \in F$. Now, since $T_1(RT_1)^{n-1}q$ is compact, we can choose, for all $k \in N$, $u_{n_k,1} \in T_1(RT_1)^{n-1}q$ such that $\|x_{n_k} - u_{n_k,1}\| = d(x_{n_k}, T_1(RT_1)^{n-1}q)$. We note that if $\{u_{n_k,1}\}$ is a subsequence of $\{u_{n,1}\}$, then $\lim_{n\to\infty} u_{n_k,1} = u \in T_1(RT_1)^{n-1}q$. By Condition (E), we have the following estimates:
From the inequality

$$\begin{aligned} \|x_{n_k} - u\| &\leq \|x_{n_k} - u_{n_k,1}\| + \|u_{n_k,1} - u\| \\ &\leq d(x_{n_k}, T_1(RT_1)^{n-1}q) + \|u_{n_k,1} - u\| \\ &\leq \mu d(x_{n_k}, T_1(RT_1)^{n-1}x_{n_k}) + \|x_{n_k} - q\| + \|u_{n_k,1} - u\|, \end{aligned} \tag{3.34}$$

and (3.21), we obtain

$$\lim_{n\to\infty} \|x_{n_k} - u\| = \|x_{n_k} - q\| \tag{3.35}$$

From Opial property of the space, we have $q = u \in T_1(RT_1)^{n-1}q$, Using similar argument as above, it is easy to see that $q \in T_i(RT_i)^{n-1}q$, for all $i = 1, 2, \cdots$,N. Therefore, $q \in \cap_{i=1}^{N} F(T_i)$.

Furthermore, we show, by mathematical induction, that for $i = 1, 2, \cdots, N$,

$$\lim_{n\to\infty} \|x_{n_k} - G_i(RG_i)^{m-1}x_{n_k}\| = 0. \tag{3.36}$$

Obviously, the conclusion is true for $m = 1$. Assume the conclusion holds for $m \geq 1$. Then, we show that it holds for all $m \in N$.

Since $\{G_i\}_{i=1}^{N}$ is uniformly L-Lipschitizian, we get, for $i = 1, 2, \cdots, N$,

$$\begin{aligned} \|x_{n_k} - G_i(RG_i)^{(m-1)+1}x_{n_k}\| &\leq \|x_{n_k} - G_i(RG_i)^{m-1}x_{n_k}\| \\ &\quad + \|G_i(RG_i)^{m-1}x_{n_k} - G_i(RG_i)^{(m-1)+1}x_{n_k}\| \\ &\leq \|x_{n_k} - G_i(RG_i)^{m-1}x_{n_k}\| + L\|x_{n_k} - G_i x_{n_k}\| \end{aligned}$$

From (3.24),(3.29) and the last inequality, we get

$$\lim_{n\to\infty} \|x_{n_k} - G_i(RG_i)^{(m-1)+1}x_{n_k}\| = 0, \quad \text{for all} \quad i = 1, 2, \cdots, N. \tag{3.37}$$

Consequently, (3.36) holds for all $m \in N$.

Thus, for each $m \in N$ and for each $x \in D$, we have (using (3.36))

$$\lim_{k\to\infty} \|x_{n_k} - x\| = \lim_{k\to\infty} \|x_{n_k} - G_i(RG_i)^{m-1}x_{n_k}\| \tag{3.38}$$

Again, since $\{G_i\}_{i=1}^{N}$ is total asymptotically nonexpansive single-valued nonself mapping, we have

$$\begin{aligned} \limsup_{k\to\infty}(\limsup_{m\to\infty} \|G_i(RG_i)^{m-1}x_{n_k} - G_i(RG_i)^{m-1}q\|) &\leq \limsup_{k\to\infty}\{\limsup_{m\to\infty}[\|x_{n_k} - q\| \\ &\quad + \mu_{m,i}\phi(\|x_{n_k} - q\|) + \xi_{m,i}]\} \\ &\leq \limsup_{k\to\infty}\{\limsup_{m\to\infty}[\|x_{n_k} - q\| \\ &\quad + \mu_{m,i}M'\|x_{n_k} - q\| + \xi_{m,i}] \\ &\leq \limsup_{k\to\infty}\{\limsup_{m\to\infty}[\|x_{n_k} - q\| \\ &\quad + \tau_m M\|x_{n_k} - q\| + \theta_m]\} \\ &\leq \limsup_{k\to\infty}\{\limsup_{m\to\infty}[(1 + \tau_m M)\|x_{n_k} - q\| \\ &\quad + \theta_m]\} \\ &= \limsup_{k\to\infty} \|x_{n_k} - q\| \end{aligned} \tag{3.39}$$

Also, from proposition 2.4 and the inequality

$$\begin{aligned} \|x_{n_k} - \frac{q + G_i(RG_i)^{m-1}q}{2}\|^2 &= \|\frac{1}{2}(x_{n_k} - q) + \frac{1}{2}(x_{n_k} - G_i(RG_i)^{m-1}q)\| \\ &\leq \frac{1}{2}\|x_{n_k} - q\| + \frac{1}{2}\|x_{n_k} - G_i(RG_i)^{m-1}q\| \\ &\quad - \frac{1}{4}g(\|q - G_i(RG_i)^{m-1}q\|), \end{aligned}$$

we have

$$\begin{aligned}\limsup_{k\to\infty}\|x_{n_k}-\frac{q+G_i(RG_i)^{m-1}q}{2}\|^2 &\le \frac{1}{2}\limsup_{k\to\infty}\|x_{n_k}-q\|^2+\frac{1}{2}\limsup_{k\to\infty}\|x_{n_k}-G_i(RG_i)^{m-1}q\|^2\\ &\quad -\frac{1}{4}g(\|q-G_i(RG_i)^{m-1}q\|)\end{aligned} \tag{3.40}$$

By the Opial property of the space and the fact that $\{x_{n_k}\}_{n\ge1}$ converges weakly to q, we get

$$\limsup_{k\to\infty}\|x_{n_k}-q\|^2=\limsup_{k\to\infty}\|x_{n_k}-\frac{q+G_i(RG_i)^{m-1}q}{2}\|^2 \tag{3.41}$$

(3.40) and (3.41) imply

$$\begin{aligned}g(\|q-G_i(RG_i)^{m-1}q\|) &\le 2\limsup_{k\to\infty}\|x_{n_k}-G_i(RG_i)^{m-1}q\|^2\\ &\quad -2\limsup_{k\to\infty}\|x_{n_k}-q\|^2\end{aligned} \tag{3.42}$$

Taking $\limsup_{m\to\infty}$ on both sides of (3.42), we have

$$\begin{aligned}\limsup_{m\to\infty}g(\|q-G_i(RG_i)^{m-1}q\|) &\le 2\limsup_{m\to\infty}(\limsup_{k\to\infty}\|x_{n_k}-G_i(RG_i)^{m-1}q\|^2)\\ &\quad -2\limsup_{k\to\infty}\|x_{n_k}-q\|^2\\ &\le 0 \quad (\,by\ (3.39))\end{aligned} \tag{3.43}$$

By the properties of g, we obtain

$$\lim_{m\to\infty}\|q-G_i(RG_i)^{m-1}q\|=0,\ for\ all\ i=1,2,\cdots,N. \tag{3.44}$$

Observe that

$$\begin{aligned}\|G_iq-q\| &\le \|G_iq-G_i(RG_i)^mq\|+\|G_i(RG_i)^mq-q\|\\ &= \|G_iq-G_i(RG_i)(RG_i)^{m-1}q\|+\|G_i(RG_i)^mq-q\|\\ &\le L\|q-(RG_i)(RG_i)^{m-1}q\|+\|G_i(RG_i)^mq-q\|\to 0\ as\ m\to\infty.\end{aligned}$$

Thus, $G_iq=q$ for all $i=1,2,\cdots,N$ and $q\in\cap_{i=1}^N F(G_i)$. Hence, $q\in F$.

Lastly, we show that $\{x_n\}_{n\ge1}$ converges weakly to q. Suppose for contradiction that there exists another subsequence $\{x_{n_i}\}_{i\ge1}$ of $\{x_n\}_{n\ge1}$ converging weakly to $p\in D$ and $p\ne q$. Following the same argument as above, we can prove that $p\in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n-q\|$ and $\lim_{n\to\infty}\|x_n-p\|$ exist. It follows from Lemma 2.3 that $q=p$. Thus, $\{x_n\}_{n\ge1}$ converges weakly to a point of F. □

Theorem 3.9. *Let E be a Banach space and D a nonempty, closed and convex subset of a real uniformly convex Banach space E. Let* $\{G_i\}_{i=1}^N : D\longrightarrow E$ *be a finite family of total asymptotically nonexpansive single-valued nonself mappings with sequences* $\{\nu_{n,i}\}_{n\ge1},\{\omega_{n,i}\}_{n\ge1}\to 0$ *as* $n\to\infty,\ for\ i=1,2,\cdots,$N. *Let* $\{T_i\}_{i=1}^N : D\longrightarrow CB(E)$ *be a type one finite family of total asymptotically quasi-nonexpansive multivalued nonself mappings from D into the family of close and bounded subsets of E with sequences* $\{\mu_{n,i}\},\{\xi_{n,i}\}\in[0,\infty):\mu_{n,i},\xi_{n,i}\to 0$ *as* $n\to\infty,\ for\ i=$

$1, 2, \cdots, \text{N}$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\left.\begin{aligned} x_1 = x \in D; \\ x_{n+1} = R(\alpha_{n,0}x_n + \textstyle\sum_{i=1}^{N} \alpha_{n,i}v_{n,i}); \\ y_n = R(\beta_{n,0}x_n + \textstyle\sum_{i=1}^{N} \beta_{n,i}G_i(RG_i)^{n-1}y_n, \forall n \in N \end{aligned}\right\} \tag{3.45}$$

where $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ *are real sequences in* $[0, 1]$, *for* $i = 1, 2, \cdots, \text{N}$, $\sum_{i=0}^{N} \alpha_{n,i} = 1 = \sum_{i=0}^{N} \beta_{n,i}$, $v_{n,i} \in T_i(RT_i)^{n-1}y_n$ *with* $\|x_n - v_{n,i}\| = d(x_n, T_i(RT_i)^{n-1}y_n)$. *Suppose* $\{T_i\}_{i=1}^{n}$ *satisfies Condition (E) and* $F = \cap_{i=1}^{N}(F(G_i) \cap F(T_i)) \neq \emptyset$ *and closed. If the following conditions hold:*

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^{\infty} \mu_{n,i} < \infty, \sum_{n=1}^{\infty} \xi_{n,i} < \infty, \sum_{n=1}^{\infty} \nu_{n,i} < \infty, \sum_{n=1}^{\infty} \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions* $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ *with* $\phi(0) = 0 = \psi(0)$ *and some constants* $M_i', M_i'' > 0$ *such that* $\psi(t) \leq M_i't$ *and* $\phi(s) = M''s$, *for* $i = 1, 2, \cdots N, \forall t, s \geq 0$.

Then, the $\{x_n\}_{n\geq1}$ *converges strongly to a point of F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. Hence, we prove only the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, F) = 0$.. From Lemma 3.1, we see that the sequence $\{x_n\}_{n\geq1}$ is monotone type (1) with respect to F. Thus, it follows by Proposition 2.5 that $\{x_n\}_{n\geq1}$converges to a point of F. This completes the proof. □

Corollary 3.10. *Let E be a Banach space and D a nonempty, compact and convex subset of a uniformly convex Banach space E. Let* $\{G_i\}_{i=1}^{N} : D \longrightarrow D$ *be an L-Lipschitizian and finite family of total asymptotically nonexpansive single-valued self mappings with sequences* $\{\nu_{n,i}\}_{n\geq1}, \{\omega_{n,i}\}_{n\geq1} \to 0$ *as* $n \to \infty$, *for* $i = 1, 2, \cdots, \text{N}$. *Let* $\{T_i\}_{i=1}^{N} : D \longrightarrow CB(D)$ *be an L-Lipschitizian and type one finite family of total asymptotically quasi-nonexpansive multivalued self mappings from D into the family of close and bounded subsets of D with sequences* $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ *as* $n \to \infty$, *for* $i = 1, 2, \cdots, \text{N}$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\left.\begin{aligned} x_1 = x \in D; \\ x_{n+1} = R(\alpha_{n,0}x_n + \textstyle\sum_{i=1}^{N} \alpha_{n,i}v_{n,i}); \\ y_n = R(\beta_{n,0}x_n + \textstyle\sum_{i=1}^{N} \beta_{n,i}G_i^n y_n, \forall n \in N \end{aligned}\right\} \tag{3.46}$$

where $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ *are real sequences in* $[0, 1]$, *for* $i = 1, 2, \cdots, \text{N}$, $\sum_{i=0}^{N} \alpha_{n,i} = 1 = \sum_{i=0}^{N} \beta_{n,i}$, $v_{n,i} \in T_i^n y_n$ *with* $\|x_n - v_{n,i}\| = d(x_n, T_i^n y_n)$. *Suppose* $\{T_i\}_{i=1}^{n}$ *satisfies Condition (E) and* $F = \cap_{i=1}^{N}(F(G_i) \cap F(T_i)) \neq \emptyset$. *If the following conditions hold:*

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^{\infty} \mu_{n,i} < \infty, \sum_{n=1}^{\infty} \xi_{n,i} < \infty, \sum_{n=1}^{\infty} \nu_{n,i} < \infty, \sum_{n=1}^{\infty} \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions* $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ *with* $\phi(0) = 0 = \psi(0)$ *and some constants* $M_i', M_i'' > 0$ *such that* $\psi(t) \leq M_i' t$ *and* $\phi(s) = M'' s$, *for* $i = 1, 2, \cdots N, \forall t, s \geq 0$.

Then, the $\{x_n\}_{n\geq 1}$ *converges strongly to a point of F*

Proof. Put $R = I$, where I is the identity mapping, then the proof follows as in Theorem 3.3. □

Corollary 3.11. *Let E be a Banach space and D a nonempty, closed and convex subset of a real uniformly convex Banach space E with the Opial property. Let* $\{G_i\}_{i=1}^N : D \longrightarrow D$ *be an L-Lipschitizian and finite family of total asymptotically nonexpansive single-valued self mappings with sequences* $\{\nu_{n,i}\}_{n\geq 1}, \{\omega_{n,i}\}_{n\geq 1} \to 0$ *as* $n \to \infty$, *for* $i = 1, 2, \cdots, N$. *Let* $\{T_i\}_{i=1}^N : D \longrightarrow KC(D)$ *be a type one finite family of total asymptotically quasi-nonexpansive multivalued self mappings from D into the family of compact convex subsets of E with sequences* $\{\mu_{n,i}\}, \{\xi_{n,i}\} \in [0, \infty) : \mu_{n,i}, \xi_{n,i} \to 0$ *as* $n \to \infty$, *for* $i = 1, 2, \cdots, N$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\left.\begin{aligned} & x_1 = x \in D; \\ & x_{n+1} = R(\alpha_{n,0} x_n + \textstyle\sum_{i=1}^N \alpha_{n,i} v_{n,i}); \\ & y_n = R(\beta_{n,0} x_n + \textstyle\sum_{i=1}^N \beta_{n,i} G_i^n y_n, \forall n \in N \end{aligned}\right\} \tag{3.47}$$

where $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ *are real sequences in* $[0, 1]$, *for* $i = 1, 2, \cdots, N$, $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$, $v_{n,i} \in T_i^n y_n$ *with* $\|x_n - v_{n,i}\| = d(x_n, T_i^n y_n)$. *Suppose* $\{T_i\}_{i=1}^n$ *satisfies Condition (E) and* $F = \cap_{i=1}^N (F(G_i) \cap F(T_i)) \neq \emptyset$ *and closed. If the following conditions hold:*

i. $0 < a \leq \alpha_{n,i}, \beta_{n,i} \leq b < 1$;

ii. $\sum_{n=1}^\infty \mu_{n,i} < \infty, \sum_{n=1}^\infty \xi_{n,i} < \infty, \sum_{n=1}^\infty \nu_{n,i} < \infty, \sum_{n=1}^\infty \omega_{n,i} < \infty$;

iii. *There exist nonnegative strictly increasing continuous functions* $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ *with* $\phi(0) = 0 = \psi(0)$ *and some constants* $M_i', M_i'' > 0$ *such that* $\psi(t) \leq M_i' t$ *and* $\phi(s) = M'' s$, *for* $i = 1, 2, \cdots N, \forall t, s \geq 0$.

Then, the $\{x_n\}_{n\geq 1}$ *converges weakly to a point of F*

Proof. Put $R = I$, where I is the identity mapping, then the proof follows as in Theorem 3.4. □

Remark 3.12. *In Theorem [3.5 and 3.6] of [37] and Theorem [3.5 and 3.7] of [36], the mappings* $\{T_i\}_{i=1}^N$ *and* $\{t_i\}_{i=1}^N$ *remained self mappings of a nonempty, closed (compact) and convex subset of a uniformly convex Banach space E into itself. Regrettably, if the domain of the operators* $\{T_i\}_{i=1}^N$ *and* $\{t_i\}_{i=1}^N$ *is a proper subset of E and* $\{T_i\}_{i=1}^N$ *and* $\{t_i\}_{i=1}^N$ *are mappings of such domain into E, the recursion formula of (1.15) and (1.16) may fail to be well defined, hence invalidating the results in [36] and [37]. To resolve this, we introduce a much more general class of nonself mappings and prove convergence results without any imposition of strict condition on the set of fixed points of T. Our results improve, extend and unify most of the existing results connected to the class of asymptotically nonexpansive single-valued and multivalued mappings. More precisely, Corollary 3.6 improves and generalizes the results of Theorem 3.5 in [36] and Theorem 3.6 in [37]. Again, Corollary 3.7 improves and generalizes results of Theorem 3.7 in [36].*

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