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A damped algorithm for minimum-norm solution of the proximal split feasibility problem

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Abstract: The aim of the paper is to introduce an iterative algorithm for finding the least norm solution of the proximal split feasibility problem. We present a damped algorithm from which strong convergence theorem is obtained in Hilbert spaces. We apply this algorithm to solve split equilibrium problem. Numerical example is included.

Keywords: Proximal split feasibility problem; damped algorithm; minimum-norm; strong convergence.

MSC: 47H10, 49M37, 49K35, 90C25.

1 Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. Finding a point x^* satisfies

$$x^* \in C \quad \text{and} \quad Ax^* \in Q \tag{1.1}$$

provided $C \bigcap A^{-1}Q \neq \emptyset$. This problem, referred to as the split feasibility problem, was introduced for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [1]. To

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solve (1.1) that has been studied extensively by many authors; see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

In order to solve (1.1), one of the key points is to use the fixed point technique according to x^* which solves (1.1) if and only if

$$x^* = \mathrm{proj}_C(I - \gamma A^*(I - \mathrm{proj}_Q)A)x^*,$$

where $\gamma > 0$ is a constant and proj_C and proj_Q stand for the orthogonal projectional on the closed convex sets C and Q, respectively. According to the above fixed point formulation, A seemingly more popular algorithm that solves (1.1) is the CQ algorithm presented by Byrne [1, 2]:

$$x_{n+1} = \mathrm{proj}_C(x_n - \gamma A^*(I - \mathrm{proj}_Q)Ax_n),$$

for all $n \geq 0$, where the initial guess $x_0 \in \mathcal{H}_1$ and $\gamma \in \left(0, \frac{2}{\lambda}\right)$, with λ being the largest eigenvalue of the matrix A^*A. Motivated by Byrne's CQ algorithm, Wang and Xu [7] introduced a modification of CQ algorithm with strong convergence and obtained the minimum-norm solution of the split feasibility problem (1.1). In particularly, they proposed the following iterative method:

$$x_{n+1} = \mathrm{proj}_C\left((1 - \alpha_n)(x_n - \gamma A^*(I - \mathrm{proj}_Q)Ax_n)\right), \tag{1.2}$$

for all $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$. Lately, Dang and Gao [8] introduced the following damped projection algorithm:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n \mathrm{proj}_C\left((1 - \alpha_n)(x_n - \gamma A^*(I - \mathrm{proj}_Q)Ax_n)\right), \tag{1.3}$$

for all $n \geq 0$, let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real numbers in $(0, 1)$, satisfying conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then the $\{x_n\}$ generated by (1.3) converges to a point in $C \bigcap A^{-1}Q \neq \emptyset$ strongly.

Our main purpose of the present manuscript is to study the more general case of the proximal split feasibility problem by introducing damped algorithm. In the sequel, $f : \mathcal{H}_1 \to \mathcal{R} \bigcup \{+\infty\}$ and $g : \mathcal{H}_2 \to \mathcal{R} \bigcup \{+\infty\}$ are two proper, lower semi-continuous convex functions. Now, we focus on the following minimization problem:

$$\min_{x \in \mathcal{H}_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.4}$$

where g_λ stands for the Moreau-Yosida approximate of the function g of index $\lambda > 0$, that is,

$$g_\lambda(x) = \min_{y \in \mathcal{H}_2} \left\{ g(y) + \frac{1}{2\lambda}\|x - y\|^2 \right\}.$$

It is worth pointing out that the split feasibility problem (1.1) is a special case of problem (1.4). As a matter of fact, we choose f and g as the indicator functions of two nonempty closed convex sets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$, respectively, that is,

$$f(x) = \delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g(x) = \delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the problem (1.4) collapses to

$$\min_{x \in \mathcal{H}_1} \left\{ \delta_C(x) + (\delta_Q)_\lambda(Ax) \right\},$$

which is equivalent to the following formulation

$$\min_{x \in C} \left\{ \frac{1}{2\lambda} \|(I - \text{proj}_Q)Ax\|^2 \right\}. \tag{1.5}$$

Surely, solving (1.5) is exactly to solve (1.1).

From (1.4) and by the Fréchet differentiability of the Yosida approximate g_λ, we have

$$\partial(f(x) + g_\lambda(Ax)) = \partial f(x) + A^* \nabla g_\lambda(Ax) = \partial f(x) + A^* \left(\frac{I - \text{prox}_{\lambda g}}{\lambda} \right) Ax, \tag{1.6}$$

where $\partial f(x)$ denotes the subdifferential of f at x and $\text{prox}_{\lambda g} x$ is the proximal mapping of g, that is,

$$\partial f(x) = \{ w \in \mathcal{H}_1 : f(y) \geq f(x) + \langle w, y - x \rangle, \forall y \in \mathcal{H}_1 \}$$

and

$$\text{prox}_{\lambda g} x = \arg\min_{y \in \mathcal{H}_2} \left\{ g(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

Note that the optimality condition of (1.6) is as follows:

$$0 \in \partial f(x) + A^* \left(\frac{I - \text{prox}_{\lambda g}}{\lambda} \right) Ax,$$

which can be rewritten as

$$0 \in \mu\lambda \partial f(x) + \mu A^* \left(I - \text{prox}_{\lambda g} \right) Ax,$$

which is equivalent to the fixed point formalation:

$$x = \text{prox}_{\mu\lambda f}(I - \mu A^*(I - \text{prox}_{\lambda g})A)x$$

for all $\mu > 0$.

If $\arg\min f \bigcap A^{-1}(\arg\min g) \neq \emptyset$, then (1.4) is reduced to the following proximal split feasibility problem of finding x such that

$$x \in \arg\min f \quad \text{and} \quad Ax \in \arg\min g, \tag{1.7}$$

where $\arg\min f = \{x^* \in \mathcal{H}_1 : f(x^*) \leq f(x), \forall x \in \mathcal{H}_1\}$ and $\arg\min g = \{x^\dagger \in \mathcal{H}_2 : g(x^\dagger) \leq g(x), \forall x \in \mathcal{H}_2\}$. In the sequel, we use Γ to denote the solution set of the problem (1.7).

Recently, in order to solve the problem (1.7), Moudafi and Thakur [13] presented the following split proximal algorithm with a way of selecting the step sizes such that its implementation does not need any prior information as regards the operator norm.

Self − adaptive split proximal algorithm [13] Let $x_0 \in \mathcal{H}_1$ be an initial arbitrarily point. Assume that a sequence $\{x_n\}$ in $\mathcal{H}_1$ has been constructed with $\theta(x_n) \neq 0$ as follows: Compute x_{n+1} via the rule

$$x_{n+1} = \text{prox}_{\mu_n \lambda f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) \tag{1.8}$$

for all $n \geq 0$, where the step size $\mu_n = \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ in which $0 < \rho_n < 4$, $h(x_n) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = \frac{1}{2}\|(I - \text{prox}_{\mu_n \lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.7) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to the sequence (1.8).

Consequently, they demonstrated the following weak convergence of the above split proximal algorithm.

Theorem 1.1. *Suppose that $\Gamma \neq \emptyset$. Assume the parameters satisfy the condition:*

$$\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n) + l(x_n)} - \epsilon$$

for some $\epsilon > 0$ small enough. Then the sequence $\{x_n\}$ generated by (1.8) *weakly converges to a solutions of the problem* (1.7).

We note that only weak convergence of algorithm (1.8) is established in [13]. However, in some applied disciplines, the strong convergence is more desirable than the weak convergence. So, Yao *et al*. [14] suggested an algorithm such that the strong convergence is guaranteed. More precisely, they presented the following scheme:

$$x_{n+1} = (1 - \alpha_n)\text{prox}_{\lambda f}\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right) \tag{1.9}$$

for all $n \geq 0$, where $\{\alpha_n\} \subset [0, 1]$ is a real number sequence. Then the $\{x_n\}$ generated by (1.9) converges strongly to a point z, which is the minimum-norm element in Γ under some mild conditons. Subsequently, Shehu [15] and Shehu and Ogbuisi [16] studied the algorithms with strong convergence for solving the proximal split feasibility problems and fixed point problems. Shehu *et al*. [17] suggested a viscosity method for the above mentioned problems.

In this paper, inspired by the recent works in this field, especially by Wang and Xu [7], Dang and Gao [8], Moudafi and Thakur [13], Yao *et al*. [14], Shehu [15], Shehu and Ogbuisi [16] and Shehu *et al*. [17], we introduce an iterative algorithm for finding the minimum-norm solution of the proximal split feasibility problem.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. The notation $Fix(T)$ denotes the set of fixed points of the mapping T, that is, $Fix(T) = \{x \in \mathcal{H} : Tx = x\}$. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Projections are an important tool for our work in this paper. Recall that the (nearest point or metric) projection [9] from $\mathcal{H}$ onto C, denoted by proj_C, is defined in such a way that, for each $x \in \mathcal{H}$, $\text{proj}_C x$ is the unique point in C with the property

$$\|x - \text{proj}_C x\| = \min\{\|x - y\| : y \in C\}.$$

Some properties of projections are gathered in the following proposition.

Propsition 2.1. *[12] Given $x \in \mathcal{H}$ and $z \in C$.*
(1) $z = \text{proj}_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$ for all $y \in C$.
(2) $z = \text{proj}_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$.
(3) $\langle x - y, \text{proj}_C x - \text{proj}_C y\rangle \geq \left\|\text{proj}_C x - \text{proj}_C y\right\|^2$ for all $y \in \mathcal{H}$, which hence implies that $proj_C$ is nonexpansive.

Definition 2.1. *[11] A nonlinear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be firmly nonexpansive if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in \mathcal{H}$, which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$$

for all $x, y \in \mathcal{H}$. Also, the mapping $I - T$ is firmly nonexpansive.

Note that the proximal mapping of g is firmly nonexpansive, namely,

$$\left\|\text{prox}_{\lambda g} x - \text{prox}_{\lambda g} y\right\|^2 \leq \langle \text{prox}_{\lambda g} x - \text{prox}_{\lambda g} y, x - y\rangle$$

for all $x, y \in \mathcal{H}_2$ and it is also the case for the complement $I - \text{prox}_{\lambda g}$.

Lemma 2.1. *[8] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \sigma_n, n \geq 0.$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ are such that
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(2) either $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\{a_n\}$ converges to zero.

To attain strong convergence result, we need to use the following lemma.

Lemma 2.2. *[10] Let $\{u_n\}$ be a sequence of real numbers. Assume $\{u_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \leq u_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : u_{n_i} < u_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$,

$$\max\{u_{\tau(n)}, u_n\} \leq u_{\tau(n)+1}.$$

3 Main result

In this section, we introduce an iterative algorithm for finding the least norm solution of the proximal split feasibility problem.

Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, $f : \mathcal{H}_1 \to \mathcal{R}\bigcup\{+\infty\}$ and $g : \mathcal{H}_2 \to \mathcal{R}\bigcup\{+\infty\}$ are two proper, lower semi-continuous convex functions and that (1.7) is consistent. $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Set $\theta(x) = \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ with $h(x) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax\|^2$, $l(x) = \frac{1}{2}\|(I - \text{prox}_{\mu_n \lambda f})x\|^2$ for all $x \in \mathcal{H}_1$ and introduce the following algorithm.

Algorithm 3.1. *For an initialization $x_1 \in \mathcal{H}_1$, assume that a sequence $\{x_n\}$ generated by the rule with $\theta(x_n) \neq 0$,*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n \text{prox}_{\mu_n \lambda f}\left((1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right)\right) \tag{3.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ are two real number sequences and μ_n is the step size satisfying $\mu_n = \frac{\rho_n(h(x_n)+l(x_n))}{\theta^2(x_n)}$ with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (1.7) and the iterative process stops. Otherwise, we set $n := n+1$ and go to the sequence (3.1).

Theorem 3.1. *Suppose that Γ is nonempty. Assume the parameters satisfy the condition:*

- *(i) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ for some $\epsilon > 0$ small enough;*
- *(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$;*
- *(iii) $0 < \liminf_{n\to\infty} \beta_n(1-\alpha_n) \leq \limsup_{n\to\infty} \beta_n(1-\alpha_n) < 1$.*

Then the sequence $\{x_n\}$ generated by (3.1) strongly converges to $x^* = \text{proj}_\Gamma 0$, which is the minimum-norm solution.

Proof. Let $x^* \in \Gamma$. Since minimizers of any function are exactly fixed points of its proximal mappings, we have $x^* = \text{prox}_{\mu_n \lambda f} x^*$ and $Ax^* = \text{prox}_{\lambda g} Ax^*$. Using the fact that $\text{prox}_{\mu_n \lambda f}$ is nonexpansive, we derive from (3.1) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
=&\left\|(1-\beta_n)x_n + \beta_n \text{prox}_{\mu_n \lambda f}\left((1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right)\right) - x^*\right\|^2 \\
=&\left\|(1-\beta_n)(x_n - x^*) + \beta_n\left(\text{prox}_{\mu_n \lambda f}\left((1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right)\right) - x^*\right)\right\|^2 \\
\leq&(1-\beta_n)\|x_n - x^*\|^2 + \beta_n\left\|\text{prox}_{\mu_n \lambda f}\left((1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right)\right) - x^*\right\|^2 \\
\leq&(1-\beta_n)\|x_n - x^*\|^2 + \beta_n\left\|(1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right) - x^*\right\|^2.
\end{aligned} \tag{3.2}$$

Note that $\nabla h(x_n) = A^*(I - \text{prox}_{\lambda g})Ax_n$ and $\nabla l(x_n) = (I - \text{prox}_{\mu_n \lambda f})x_n$. Since $I - \text{prox}_{\lambda g}$ is firmly nonexpansive. Hence, we obtain

$$
\begin{aligned}
&\|x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n - x^*\|^2 \\
=&\|x_n - x^*\|^2 - 2\mu_n \left\langle A^*(I-\text{prox}_{\lambda g})Ax_n, x_n - x^* \right\rangle + \mu_n^2 \|A^*(I-\text{prox}_{\lambda g})Ax_n\|^2 \\
\leq&\|x_n - x^*\|^2 - 4\mu_n h(x_n) + \mu_n^2 \|\nabla h(x_n)\|^2 \\
\leq&\|x_n - x^*\|^2 - \frac{4\rho_n (h(x_n)+l(x_n))^2}{\theta^2(x_n)} \frac{h(x_n)}{h(x_n)+l(x_n)} + \frac{\rho_n^2 (h(x_n)+l(x_n))^2}{\theta^2(x_n)} \\
=&\|x_n - x^*\|^2 - \frac{\rho_n (h(x_n)+l(x_n))^2}{\theta^2(x_n)} \left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n \right).
\end{aligned}
\tag{3.3}
$$

Without loss of generality, by the control condition (i), we can assume that $(4h(x_n))/(h(x_n)+l(x_n)) - \rho_n \geq 0$ for all $n \geq 1$. Thus, from (3.2), (3.3) and the conditions (iii), we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq&(1-\beta_n)\|x_n - x^*\|^2 + \beta_n \left\| (1-\alpha_n)\left(x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n \right) - x^* \right\|^2 \\
\leq&(1-\beta_n)\|x_n - x^*\|^2 + \beta_n (1-\alpha_n) \left\| x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n - x^* \right\|^2 + \alpha_n \beta_n \|x^*\|^2 \\
\leq&(1-\beta_n)\|x_n - x^*\|^2 + \beta_n (1-\alpha_n) \|x_n - x^*\|^2 + \alpha_n \beta_n \|x^*\|^2 \\
\leq&\max \left\{ \|x_n - x^*\|^2, \|x^*\|^2 \right\},
\end{aligned}
\tag{3.4}
$$

which yields that sequence $\{x_n\}$ is bounded.

Returning to (3.3) and (3.4), we have

$$
\begin{aligned}
&\beta_n (1-\alpha_n) \left(\frac{\rho_n (h(x_n)+l(x_n))^2}{\theta^2(x_n)} \left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n \right) \right) \\
\leq&(1-\alpha_n\beta_n)\|x_n - x^*\|^2 + \alpha_n \beta_n \|x^*\|^2 - \|x_{n+1} - x^*\|^2.
\end{aligned}
\tag{3.5}
$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq m$. In this case, we know that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This together with (3.5), conditions $(i - iii)$ and $\rho_n \left(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n \right) \geq \epsilon^2$ implies that

$$
\lim_{n\to\infty} \frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} = 0. \tag{3.6}
$$

Noting that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded, we deduce from (3.6) immediately that

$$
\lim_{n\to\infty} (h(x_n) + l(x_n)) = 0.
$$

Therefore, we have

$$
\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0. \tag{3.7}
$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. By the lower semi-continuity of h, we have

$$
0 \leq h(p) \leq \liminf_{i\to\infty} h(x_{n_i}) = \lim_{n\to\infty} h(x_n) = 0,
$$

so, we have

$$h(p) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ap\| = 0,$$

that is, Ap is a fixed point of proximal mapping of g or, equivalently, $0 \in \partial g(Ap)$. In other words, Ap is a minimizer of g.

Similarly, from the lower semi-continuity of l, we have

$$0 \leq l(p) \leq \liminf_{i\to\infty} l(x_{n_i}) = \lim_{n\to\infty} l(x_n) = 0,$$

so, we have

$$l(p) = \frac{1}{2}\|(I - \text{prox}_{\mu_n \lambda f})p\| = 0,$$

that is, p is a fixed point of proximal mapping of f or, equivalently, $0 \in \partial f(p)$. In other words, p is a minimizer of f. Hence, $p \in \Gamma$.

Let $z_n = (1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right)$, $n \geq 1$. Then

$$\begin{aligned}\|z_n - x_n\| =& \|(1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right) - x_n\| \\ \leq& (1-\alpha_n)\mu_n\|A^*(I - \text{prox}_{\lambda g})Ax_n\| + \alpha_n\|x_n\|.\end{aligned}$$

We observe that $0 < \mu_n < \frac{4(h(x_n)+l(x_n))}{\theta^2(x_n)}$, which implies that $\lim_{n\to\infty}\mu_n = 0$. Furthermore, we obtain from the condition (ii) and the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

This together with $x_{n_i} \rightharpoonup p$ implies that $z_{n_i} \rightharpoonup p$. Therefore, we have

$$\limsup_{n\to\infty}\langle x^*, z_n - x^*\rangle = \lim_{i\to\infty}\langle x^*, z_{n_i} - x^*\rangle = \langle x^*, p - x^*\rangle \geq 0. \tag{3.8}$$

Thus, it follows from (3.2) and (3.3) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ \leq& (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\left\|(1-\alpha_n)\left(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n\right) - x^*\right\|^2 \\ \leq& (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\left((1-\alpha_n)\|x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n - x^*\|^2 + 2\langle\alpha_n(-x^*), z_n - x^*\rangle\right) \\ \leq& (1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\langle -x^*, z_n - x^*\rangle.\end{aligned} \tag{3.9}$$

Applying Lemma 2.1 and (3.8) to (3.9), we deduce $x_n \to x^*$.

Case 2. Assume there exists an integers n_0, such that

$$\|x_{n_0} - x^*\| \leq \|x_{n_0+1} - x^*\|.$$

Set $u_n = \{\|x_n - x^*\|\}$, then we have

$$u_{n_0} \leq u_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{l \geq 1 : n_0 \leq l \leq n, u_l \leq u_{l+1}\}.$$

It is clear that $\tau(n)$ is non-decresing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and

$$u_{\tau(n)} \leq u_{\tau(n)+1}$$

for all $n \geq n_0$.

By a similar argument to that of Case 1, we can obtain that

$$\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(z_{\tau(n)}) \subset \Gamma.$$

Thus, we obtain

$$\limsup_{n\to\infty} \langle x^*, z_{\tau(n)} - x^* \rangle \geq 0. \tag{3.10}$$

Since $u_{\tau(n)} \leq u_{\tau(n)+1}$, we have from (3.9) that

$$u^2_{\tau(n)} \leq u^2_{\tau(n)+1} \leq (1 - \alpha_{\tau(n)}\beta_{\tau(n)})u^2_{\tau(n)} + 2\alpha_{\tau(n)}\beta_{\tau(n)}\langle -x^*, z_{\tau(n)} - x^* \rangle. \tag{3.11}$$

It follows that

$$u^2_{\tau(n)} \leq 2\langle -x^*, z_{\tau(n)} - x^* \rangle. \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$\limsup_{n\to\infty} u_{\tau(n)} \leq 0$$

and hence

$$\lim_{n\to\infty} u_{\tau(n)} = 0. \tag{3.13}$$

By (3.11), we obtain

$$\limsup_{n\to\infty} u^2_{\tau(n)+1} \leq \limsup_{n\to\infty} u^2_{\tau(n)}.$$

This together with (3.13) implies that

$$\lim_{n\to\infty} u_{\tau(n)+1} = 0.$$

Applying Lemma 2.2 to get

$$0 \leq u_n \leq \max\{u_{\tau(n)}, u_{\tau(n)+1}\}.$$

therefore, $u_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

Remark 3.1. *We make the following remark concerning our contributions in this paper.*

- *To ensure the weak convergence of the algorithm proposed by Moudafi and Thakur in [[13], Theorem 2.2], one has to use the Opial's property of Hilbert space. The main advantage of our algorithm is that its convergence does not rely on the Opial's property. Furthermore, we establish strong convergence of the proposed algorithm to the minimum-norm solution of the proximal split feasibility problem.*
- *If we take $f = \delta_C$ and $g = \delta_Q$, then it is clear that Algorithm 3.1 collapses to the Algorithm 3.1 in Dang and Gao [8]. Moreover, we get the strong convergence of the proposed algorithm to the minimum-norm solution of the proximal split feasibility problem. This is another interesting point.*
- *If we take $f = \delta_C$, $g = \delta_Q$ and $\beta_n = 1$, Algorithm 3.1 reduces to the Algorithm 4.1 in Wang and Xu [7].*
- *Theorem 3.1 extends the corresponding results of Dang and Gao [[8], Theorem 3.1] and Wang and Xu [[7], Theorem 4.3] by discarding the assumption $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$.*

4 Generalization

Indeed, convex optimization is a special case of monotone equilibrium problem. We next suggest an extension of our results to split equilibrium problem.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ and g be a bifunction of $Q \times Q$ into $\mathbb{R}$. Let A: $\mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint A^*. The split equilibrium problem is to find a point x^* such that

$$f(x^*, y) \geq 0, \quad \forall y \in C \tag{4.1a}$$

$$g(Ax^*, y) \geq 0, \quad \forall y \in Q. \tag{4.1b}$$

We shall denote the solutions set of (4.1a) and (4.1b) by Γ. In order to solve the split equilibrium problem for f and g, let us assume that f and g satisfy the following conditions:

- (C1) $f(x, x) = 0$ for all $x \in C$ and $g(y, y) = 0$ for all $y \in Q$;
- (C2) f and g are monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$ and $g(x, y) + g(y, x) \leq 0$ for all $x, y \in Q$
- (C3) for each $x, y \in C$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$ and for each $x, y \in Q$, $\lim_{t\to 0} g(tz + (1-t)x, y) \leq g(x, y)$;
- (C4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous and for each $x \in Q$, $y \mapsto g(x, y)$ is convex and lower semicontinuous.

Lemma 4.1. *[18] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be two nonempty closed convex sets. Assume that f, g satisfy $(C1) - (C4)$. For $r > 0$, define mappings $T_r^f : \mathcal{H}_1 \to C$ and $T_r^g : \mathcal{H}_2 \to Q$ as follows:*

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

and

$$T_r^g(x) = \left\{ z \in Q : g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in Q \right\}.$$

Then, we have the following assertions.

- *(1) T_r^f and T_r^g are single-valued;*
- *(2) T_r^f and T_r^g are firmly nonexpansive mapping;*
- *(3) $Fix(T_r^f) = EP(f)$ and $Fix(T_r^g) = EP(g)$;*
- *(4) $EP(f)$ and $EP(g)$ are closed and convex.*

We are now in a position to solve the split equilibrium problem (4.1a) and (4.1b).

Theorem 4.1. *Let f be a bifunction from $C \times C$ and g a bifunction from $Q \times Q$ both satisfying $(C1) - (C4)$ such that Γ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1]. Assume the parameters satisfy the condition:*

- *(i) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ for some $\epsilon > 0$ small enough;*
- *(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$;*
- *(iii) $0 < \liminf_{n\to\infty} \beta_n(1-\alpha_n) \leq \limsup_{n\to\infty} \beta_n(1-\alpha_n) < 1$.*

Then the sequence $\{x_n\}$ generated by $x_1 \in C$;

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T_r^f\left((1-\alpha_n)\left(x_n - \mu_n A^*(I - T_r^g)Ax_n\right)\right)$$

strongly converges to $x^* = \mathrm{proj}_\Gamma 0$, which is the minimum-norm solution.

Proof. Replace the proximal mappings of the convex functions f and g in Theorem 3.1 by the resolvent operators associated to the two monotone equilibrium bifunctions T_r^f and T_r^g. Hence, we have the desired result. □

5 Numerical result

In this section, we provide concrete example including numerical result of the problem considered in Section 3 of this paper. All codes were written in Matlab R2014a(8.3.0.532) and run on Lenovo i5 Dual Core 8G RAM laptop.

Numerical Example Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$ with the standard norm $|\cdot| = \|\cdot\|$ for all $x, y \in \mathbb{R}$. Let $C = Q = [0, +\infty)$ and take $Ax = 10x$ for all $x \in \mathbb{R}$. In problem (1.7), let $f = \delta_C$ and $g = \delta_Q$ be the indicator functions of two nonempty closed convex sets C, Q of $\mathbb{R}$, respectively. Then

$$\mathrm{prox}_{\mu_n\lambda f}(x) = P_C(x) = \mathrm{prox}_{\lambda g}(x) = P_Q(x) = \begin{cases} x, & x \in [0, +\infty); \\ 0, & x \notin [0, +\infty). \end{cases}$$

Hence, problem (1.7) become: Find some point x in C such that $Ax \in Q$. Now, let $\rho_n = 2$, $\alpha_n = \frac{11}{50}$, $\beta_n = \frac{3}{35}$. Also, $h(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$, $l(x_n) = \frac{1}{2}\|(I - P_C)x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ with $\nabla h(x_n) = A^*(I - P_Q)Ax_n$, $\nabla l(x_n) = (I - P_C)x_n$.

It can be observed that all the assumptions of Theorem 3.1 are satisfied. We now rewrite (3.1) as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C\left((1 - \alpha_n)\left(x_n - \mu_n A^*(I - P_Q)Ax_n\right)\right), \quad n \geq 1,$$

Choosing initial values $x_1 = 6$ and $x_1 = -3$, respectively, we see that figures and numerical results demonstrate Theorem 3.1.

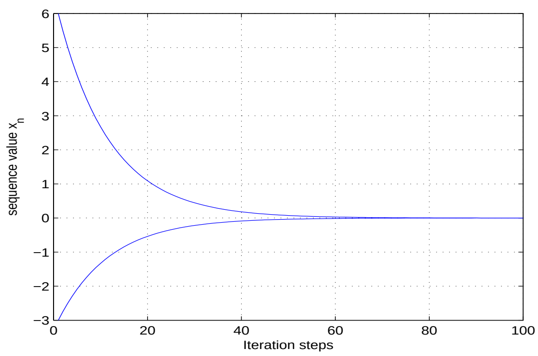

Figure 1: The convergence of $\{x_n\}$ with initial values 6 and -3, respectively.

n	x_n	x_n
1	6.000000000000000	-3.000000000000000
2	5.485714285714286	-2.740871684260145
3	5.015510204081632	-2.504125863193015
4	4.585609329446064	-2.287829224083077
5	4.192557101207830	-2.090215446237388
...	...	...
97	0.001101582204421	-0.000513802730152
98	0.001007160872614	-0.000469422451457
99	0.000920832797818	-0.000428875568385
100	0.000841904272291	-0.000391830967152

Table 1: The values of the sequence $\{x_n\}$

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