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The existence and uniqueness of $\varphi-$Best proximity point theorems for generalized Boyd-Wong proximal contraction

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Abstract: In this paper, we study concept and expand the condition of Işik, Sezen and Vetro [11] to prove the existence and uniqueness of $\varphi-$best proximity point for $(F, \lambda, \varphi)-$weak proximal and $(F, \lambda, \varphi)-$proximal contraction. As for applications of our main results, we prove the existence and uniqueness of a fixed point on partial metric spaces and variational inequality.

Keywords: $(F, \lambda, \varphi)-$proximal contraction, $(F, \lambda, \varphi)-$weak proximal contraction, $\varphi-$best proximity point, partial metric space.

MSC: 47H10,54H25.

1 Introduction

In 1969 Boyd and Wong [5] introduced the well-known theorems of Banach which states that, if X is a complete metric space and $T : X \to X$ satisfies $d(Tx, Ty) \leq kd(x, y)$ that constant k replaced by some control function, then T has a unique fixed point.

Now, we introduce the concept of $\varphi-$fixed point by given that (X, d) a metric space, $\varphi : X \to [0, \infty)$ and $T : X \to X$ be an operator on X. The set of all fixed points of the operator T is denoted by $F_T = \{x \in$

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$X : x = Tx\}$ and we said $Z_\varphi = \{x \in X : \varphi(x) = 0\}$ be the set of zeros of the function φ. The element $z \in X$ is called a φ-fixed point of the operator T if and only if $z \in F_T \cap Z_\varphi$.

In 2014 Jleli M, Samet B and Vetro C [12] introduced the idea of φ mapping and illustrated some $\varphi-$fixed point for the various classes of operators which defined on a metric space (X, d).

Suppose (X, d) be a metric space, A, B is a nonempty subsets of X and $T : A \to B$. Then $x \in A$ is called a best proximity point of T if the follwings problem,

$$d(x, Tx) = d(A, B). \tag{1.1}$$

From (1.1) if $Tx = x$ then it will be transfer to the fixed point problem.

Later then, Işik, Sezen and Vetro [11] introduced the notion of $\varphi-$best proximity point (we define in the section 2) and give two contraction include $(F, \varphi)-$proximal and weak proximal contraction prove to the existence and uniqueness of $\varphi-$best proximity point.

In this paper, we study concept and expand the condition of Işik, Sezen and Vetro [11] to prove the existence and uniqueness of $\varphi-$best proximity point by replaced constants k by a self-mapping $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(t) < t$ for $t > 0, \lambda(0) = 0$ and $\limsup_{s\to t^+} \lambda(s) < t$ for all $t > 0$, which generalization the result. As for applications of our main results, we prove the existence and uniqueness of a fixed point on partial metric spaces and variational inequality.

2 Preliminaries and Auxiliaries

In this section, we introduce the definitions necessary to prove in section 3 as follows.

Given that (X, d) be a metric space, $T : X \to X$ be a self-mapping and $F_T = \{x \in X : Tx = x\}$ is called the set of all fixed points of T. $Z_\varphi = \{x \in X : \varphi(x) = 0\}$ when $\varphi : X \to [0, \infty)$. Moreover, $\mathcal{F}$, denote the collection of functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F1) $\max\{u, v\} \le F(u, v, w)$ for all $u, v, w \in [0, \infty)$;

(F2) $F(u, 0, 0) = a$ for all $u \ge 0$;

(F3) F is continuous.

Next, we will give some example of F.

(1) $F(a, b, c) = a + b + c, \forall a, b, c \in [0, \infty)$;

(2) $F(a, b, c) = \max\{a, b\}, \forall a, b, c \in [0, \infty)$;

(3) $F(a, b, c) = a + b^2 + c, \forall a, b, c \in [0, \infty)$.

Next, we move to consider non-self mapping on X be let (X, d) be a metric space, A and B be nonempty subsets of X. The following notions are used in this work.

$$\begin{aligned} d(A, B) &:= \inf\{d(x, y) : x \in A, y \in B\}; \\ A_0 &:= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}; \\ B_0 &:= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

For any non-self mapping $T : A \to B$, the set of all best proximity points of T is denoted by

$$B_{est}(T) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

$Z_\varphi = \{x \in A : \varphi(x) = 0\}$ is called set of zeros of $\varphi : A \to [0, \infty)$. Next, We recommend the following important definitions.

Definition 2.1. Let $T : A \to B$ be non-self mapping, for each $x^* \in A$ if $x^* \in B_{est}(T) \cap Z_\varphi$ then x^* is said to be $\varphi-$best proximity point of T.

Definition 2.2. Let (X, d) be a metric space, A, B are nonempty subsets of X, with $F \in \mathcal{F}$, $\varphi : A \to [0, \infty)$ and $\lambda : [0, \infty) \to [0, \infty)$ be two given self-mappings such that $\lambda(t) < t$ for $t > 0, \lambda(0) = 0$ and $\limsup_{s \to t^+} \lambda(s) < t$ for all $t > 0$. Then, $T : A \to B$ is an $(F, \lambda, \varphi)-$weak proximal contraction, if there exists $M \geq 0$ such that

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases}$$

$$\Rightarrow F(d(u, v), \varphi(u), \varphi(v)) \leq \lambda(F(d(x, y), \varphi(x), \varphi(y))) + M(F(d(y, u), \varphi(y), \varphi(u)) - F(0, \varphi(y), \varphi(u))) \quad (2.1)$$

for all $v, u, x, y \in A$

3 Main Results

In this section, we establish the existence of $\varphi-$best proximity point for $(F, \lambda, \varphi)-$weak proximal contraction. Afterwards, we prove the uniqueness of $\varphi-$best proximity point. By adding the assumption $(H5)$ and using $(F, \lambda, \varphi)-$proximal contraction, we obtain the uniqueness of $\varphi-$best proximity point. Let (X, d) be a metric space. A and B be two nonempty subsets of X. We modify the following assumption for using in our main result.

(H1) nonempty set A_0 is complete with respect to the topology induced by d;

(H2) $T(A_0) \subseteq B_0$;

(H3) $\varphi : A \to [0, \infty)$ is lower semi-continuous;

(H4) $T : A \to B$ is a $(F, \lambda, \varphi)-$weak proximal contraction.

Theorem 3.1. *Assume that $(H1) - (H4)$ hold, then there exists $x^* \in B_{est}(T)$ such that $B_{est}(T) \cap Z_\varphi$ be a nonempty set Forthermore, for any $x \in X$, the sequence $\{T^n x\}$ converges to a $\varphi-$best proximity point of T.*

Proof. From $(H1)$ and $(H2)$, there exists element x_0 in A_0 with $Tx_0 \in T(A_0) \subseteq B_0$ we get $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Similarly, since $Tx_1 \in T(A_0) \subseteq B_0$. We have x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$. Recursively the process, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B), \ \forall n \in \mathbb{N}. \quad (3.1)$$

We can see that, if there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

Next, we shows that x_{n_0} is a best proximity point of T and then the proof is completed. So, we suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.
We set

$$a_n := F(d(x_n, x_{n+1}), \varphi(x_n,)\varphi(x_{n+1})).$$

By (3.1),(2.1) and since $\lambda(t) < t$ for each $t > 0$, then we see that

$$\lambda(a_{n+1}) < a_{n+1} \leq \lambda(a_n) < a_n,$$

for all $n \in \mathbb{N}$. It concluded that $\{a_n\}$ and $\{\lambda(a_n)\}$ is a strictly decreasing sequence of non-negative real number and bounded below. Therefore $\lim_{n\to\infty} a_n$ exists. Assume that

$$a := \lim_{n\to\infty} a_n \geq 0.$$

We claim that $a = 0$. Suppose for the sake of contradiction that $a > 0$. If $\limsup_{s\to t^+} \lambda(s) < t, \forall t > 0$, then we obtain that $\limsup_{t_n\to a^+} \lambda(t_n) < a$ whenever $t_n \to a^+$ as $n \to +\infty$. That is

$$\begin{aligned} 0 < a &= \lim_{n\to+\infty} a_{n+1} \\ &\leq \lim_{n\to+\infty} \lambda(a_n) \\ &\leq \limsup_{s\to a^+} \lambda(s) \\ &< a, \end{aligned}$$

which is a contradiction. Thus, we must have $\lim_{n\to+0} a_n = 0$. Now using the condition $(F1)$, we get

$$0 \leq \varphi(x_n) \leq \max\{d(x_n, x_{n+1}), \varphi(x_n)\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n$$

$$0 \leq d(x_n, x_{n+1}) \leq \max\{d(x_n, x_{n+1}), \varphi(x_n)\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above two inequalities, then we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} \varphi(x_n) = 0, \text{ for each } x \in X. \tag{3.2}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence by assuming that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k > n_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \varepsilon, \forall n \in \mathbb{N}$. Then, we can choose m_k to be satisfied $d(x_{m_k-1}, x_{n_k}) < \varepsilon$. For each $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon \end{aligned}$$

we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.3}$$

By using (3.2),(3.3),$(F2)$ and the continuity of F, we will get

$$\lim_{k\to\infty} F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) = F(\varepsilon, 0, 0) = \varepsilon. \tag{3.4}$$

Then by using the contractive condition (2.1), $(F1)$ and the triangle inquality, we obtain, $\forall k \in \mathbb{N}$:

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\le d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k+1}) + \max\{d(x_{m_k+1}, x_{n_k+1}) + \varphi(x_{m_k+1})\} + d(x_{n_k+1}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k+1}) + F(d(x_{m_k+1}, x_{n_k+1}), \varphi(x_{m_k+1}), \varphi(x_{n_k+1})) + d(x_{n_k+1}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k+1}) + \lambda(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k}))) + d(x_{n_k+1}, x_{n_k}). \end{aligned}$$

For all $k \in \mathbb{N}$. Taking $k \to \infty$ in the above inequality, using (3.2), (3.4), $\limsup_{s\to t^+} \lambda(s) < t, \forall t > 0$ and the continuity of F and $F_k := F(d(x_{m_k}, x_{n_k}), \phi(x_{m_k}), \phi(x_{n_k}))$ with $\lim_{k\to\infty} F_k = \varepsilon$, we get

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \\ &\le \limsup_{k\to\infty} \lambda(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k}))) \\ &= \limsup_{k\to\infty} \lambda(F_k) \\ &\le \limsup_{s\to\varepsilon^+} \lambda(s) \\ &< \varepsilon \end{aligned}$$

which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence in X. Since A_0 is complete, there exists $x^* \in A_0$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{3.5}$$

Now, we show that x^* is a $\varphi-$best proximity point of T. By (3.2), (3.5) and φ is lower semi-continuous, we will get

$$\varphi(x^*) = 0. \tag{3.6}$$

And, since $x^* \in A_0$. By using $(H2)$, we have $z \in A_0$ such that

$$d(z, Tx^*) = d(A, B). \tag{3.7}$$

By $(H4)$, (3.1), (3.7) we obtain

$$\begin{aligned} F(d(x_{n+1}, z), \varphi(x_{n+1}), \varphi(z)) \le\ & \lambda(F(d(x_n, x^*), \varphi(x_n), \varphi(x^*))) + M[F(d(x_{n+1}, x^*), \varphi(x_{n+1}), \varphi(x^*)) \\ & - F(0, \varphi(x_{n+1}), \varphi(x^*))]. \end{aligned}$$

Taking the limit as $n \to \infty$ and use (3.5), (3.2), (3.6) and the continuity of F, we have

$$F(d(x^*, z), 0, \varphi(z)) \le \lambda(F(0, 0, 0)).$$

By $(F2)$, we get

$$F(d(x^*, z), 0, \varphi(z)) \le 0.$$

From conndition $(F1)$, we obtain $d(x^*, z) = 0$ and so $x^* = z$. By (3.7), we have

$$d(x^*, Tx^*) = d(A, B).$$

Imply that x^* is a $\varphi-$best proximity point of T. □

Example 3.2. Let $X = [0,1] \cup \{2,3,4,...\}$ and $d : X \times X \to \mathbb{R}$ be defined by $d((x,i),(y,j)) = \sqrt{d_x^2(x,y) + |i-j|^2}$ when

$$d_x(x,y) = \begin{cases} |x-y| & , \text{ if } x,y \in [0,1] \\ x+y & , \text{ if } x \text{ or } y \notin [0,1],\ x \neq y \\ 0 & , \text{ if } x \text{ or } y \notin [0,1],\ x = y. \end{cases}$$

Let $A = X \times \{0\} = A_0$, $B = X \times \{1\} = B_0$ and $d(A,B) = 1$. Let $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = a+b$, $T : A \to B$ be defined by

$$T(x,0) = \begin{cases} (x - \frac{1}{2}x^2, 1) & , x,y \in [0,1] \\ (x-1, 1) & , x \notin [0,1] \end{cases}$$

and $\lambda : X \to [0,\infty)$ defined by

$$\lambda(t) = \begin{cases} t - \frac{1}{2}t^2 & , t \in [0,1] \\ t-1 & , t > 1. \end{cases}$$

Finally, defined

$$\varphi(x) = \begin{cases} 1_{X\cap[2,\infty)} \\ 0_{x\in[0,1]} \end{cases}$$

Proof. We set $M = 1$

Case 1 If $x,y \in [0,1] \Rightarrow u,v \in [0,1]$

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d\left((x-\frac{1}{2}x^2,0),(y-\frac{1}{2}y^2)\right) \\ &= \left|(x-\frac{1}{2}x^2)-(y-\frac{1}{2}y^2)\right| \end{aligned}$$

$$\begin{aligned} \lambda(F(d(x,y),\varphi(x),\varphi(y))) + d(y,Tx) &= \lambda(d((x,0),(y,0))) + d((y,0),(x-\frac{1}{2}x^2,1)) \\ &= \lambda(|x-y|) + \sqrt{(y-x+\frac{1}{2}x^2)+1} \\ &= \left(|x-y| - \frac{1}{2}(x-y)^2\right) + \sqrt{(y-x+\frac{1}{2}x^2)+1}. \end{aligned}$$

Hence $(x-\frac{1}{2}x^2)-(y-\frac{1}{2}y^2) \leq \left(|x-y|-\frac{1}{2}(x-y)^2\right) + \sqrt{(y-x+\frac{1}{2}x^2)+1}$.

Case 2 If $x,y \notin [0,1]$, $x \neq y$

If $x=(2,0)$ then $y>2$, we will get

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d((1,0),(y-1,0)) \\ &= \sqrt{(1+y-1)^2+0} = \sqrt{y^2} = y \\ \text{and } F(d(u,v),\varphi(v),\varphi(u)) &= d((1,0),(y-1,0))+1 \\ &= \sqrt{(1+y-1)^2+0}+1 = y+1 \end{aligned}$$

$$\begin{aligned} \lambda(F(d(x,y),\varphi(x),\varphi(y)))+d(y,Tx) &= \lambda(d((2,0),(y,0))+1)+d((y,0),(1,1)) \\ &= \lambda(y+3)+\sqrt{(y+1)^2+1} \\ &= (y+2)+\sqrt{(y+1)^2+1}. \end{aligned}$$

Hence $y+1<(y+2)+\sqrt{(y+1)^2+1}$.

If $x\neq y\neq(2,0)$

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d((x-1,0),(y-1,0))+1 \\ &= x+y-1 \end{aligned}$$

$$\begin{aligned} \lambda(F(d(x,y),\varphi(x),\varphi(y)))+d(y,Tx) &= \lambda(d((x,0),(y,0))+1)+d((y,0),(x-1,1)) \\ &= \lambda(x+y+1)+\sqrt{(x+y-1)^2+1} \\ &= (x+y)+\sqrt{(x+y-1)^2+1}. \end{aligned}$$

Hence $(x+y-1)\leq(x+y)+\sqrt{(x+y-1)^2+1}$.

Case 3 If $x\in[0,1],\ y\notin[0,1]$

If $y=(2,0)$, we get

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d((x-\frac{1}{2}x^2,0),(1,0)) \\ &= (x+1-\frac{1}{2}x^2) \end{aligned}$$

$$\begin{aligned} \lambda(F(d(x,y),\varphi(x),\varphi(y)))+d(y,Tx) &= \lambda(d((x,0),(2,0)))+d((2,0),(x-\frac{1}{2}x^2,1)) \\ &= \lambda(x+2)+\sqrt{(2+x-\frac{1}{2}x^2)^2+1} \\ &= (x+y)+\sqrt{(2+x-\frac{1}{2}x^2)^2+1}. \end{aligned}$$

If $y>(2,0)$, we get

$$\begin{aligned} F(d(u,v),\varphi(u),\varphi(v)) &= d((x-1,0),(y-1,0))+1 \\ &= x+y-1 \end{aligned}$$

$$\begin{aligned} \lambda(F(d(x,y),\varphi(x),\varphi(y)))+d(y,Tx) &= \lambda(d((x-\frac{1}{2}x^2,0),(y-1,0)))+d((y,0),(x-\frac{1}{2}x^2,1)) \\ &= \lambda(x+y)+\sqrt{(y+x-\frac{1}{2}x^2)^2+1} \\ &= (x+y-1)+\sqrt{(y+x-\frac{1}{2}x^2)^2+1} \\ \text{and } \lambda(F(d(x,y),\varphi(y),\varphi(x)))+d(y,Tx) &= (x+y)+\sqrt{(y+x-\frac{1}{2}x^2)^2+1}. \end{aligned}$$

□

Next, we can prove uniqueness of theorem 3.1 by similar establishing to the case of an $(F,\lambda,\varphi)-$weak proximal contraction.

$(H5)$ There exist $M \geq 0$ such that

$$\begin{cases} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases}$$

$$\Rightarrow F(d(u,v),\varphi(u),\varphi(v)) \leq \lambda(F(d(x,y),\varphi(x),\varphi(y))) + M(F(d(x,u),\varphi(x),\varphi(u)) - F(0,\varphi(x),\varphi(u))) \quad (3.8)$$

$\forall u,v,x,y \in A$

Theorem 3.3. *Suppose that $(H1)-(H5)$ hold, then there exists $x^* \in A$ is a unique solution such that $B_{est}(T) \cap Z_\varphi = \{x^*\}$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} T^n x = x^*$.*

Proof. We prove the uniqueness of $\varphi-$best proximity point. From theorem 3.1, there exists x^* be a $\varphi-$best proximity point of T. Assume that ς is also $\varphi-$best proximity point of T, then we have

$$d(x^*,Tx^*) = d(\varsigma,T\varsigma) = d(A,B),\ \varphi(x^*) = 0, \text{ and } \varphi(\varsigma) = 0.$$

By $(H5)$, we get

$$\begin{aligned} F(d(x^*,\varsigma),\varphi(x^*),\varphi(\varsigma)) &\leq \lambda(F(d(x^*,\varsigma),\varphi(x^*),\varphi(\varsigma))) \\ &\quad +M[F(d(x^*,x^*),\varphi(x^*),\varphi(x^*)) - F(0,\varphi(x^*),\varphi(x^*))] \\ &= \lambda(F(d(x^*,\varsigma),\varphi(x^*),\varphi(\varsigma))) \\ \Rightarrow F(d(x^*,\varsigma),0,0) &\leq \lambda(F(d(x^*,\varsigma),0,0)). \end{aligned}$$

We conclude that $d(x^*,\varsigma) = 0$ which means that $x^* = \varsigma$. Hence, x^* is a unique $\varphi-$best proximity point of T. □

Next results $M = 0$ then our contraction reduce to $(F,\lambda,\varphi)-$proximal as follows:

Definition 3.4. Let (X,d) be a metric space, A and B be two nonempty subsets of X with $F \in \mathcal{F}$, $\varphi : A \to [0,\infty)$ and $\lambda : [0,\infty) \to [0,\infty)$ be two given funtions such that $\lambda(t) < t$ for $t > 0, \lambda(0) = 0$ and $\limsup_{s\to t^+} \lambda(s) < t$ for all $t > 0$. Then, $T : A \to B$ is an $(F,\lambda,\varphi)-$proximal contraction, if

$$\begin{cases} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Rightarrow F(d(u,v),\varphi(u),\varphi(v)) \le \lambda(F(d(x,y),\varphi(x),\varphi(y))) \tag{3.9}$$

for all $u,v,x,y \in A$

Theorem 3.5. *Suppose that* $(H1)-(H3)$ *and* (3.4) *hold, then there exists* $x^* \in A$ *is a unique solution such that* $B_{est}(T)\cap Z_\varphi = \{x^*\}$. *Furthermore, for any* $x \in X$ *then we have* $\lim_{n\to\infty} T^n x = x^*$.

Proof. From Theorem 3.1, denote by $M = 0$, we have x^* is a $\varphi-$best proximity point of T. Now, we show that $B_{est}(T)\cap Z_\varphi$ is a singleton. Assome that $\varsigma \in A$ is another $\varphi-$best proximity point of T, so, we have

$$d(x^*,Tx^*) = d(\varsigma,T\varsigma) = d(A,B),\ \varphi(x^*) = 0, \text{ and } \varphi(\varsigma) = 0.$$

From (3.9) we obtain

$$\begin{aligned} F(d(x^*,\varsigma),\varphi(x^*),\varphi(\varsigma)) &\le \lambda(F(d(x^*,\varsigma),\varphi(x^*),\varphi(\varsigma))). \\ \text{Then, we have } F(d(x^*,\varsigma),0,0) &\le \lambda(F(d(x^*,\varsigma),0,0)), \end{aligned}$$

which implics that $d(x^*,\varsigma) = 0$, so we get $x^* = \varsigma$. Therefore, x^* is a unique $\varphi-$best proximity point of T. □

4 Applications

In this section, we prove result in partial metric space and variational inequality problem by theorem 3.1

4.1 Some consequences on partial metric space

Definition 4.1. Matthews [15]. A partial metric on a nonempty set X is a function $p : X\times X \to [0,\infty)$ such that for all $x,y,z \in X$:

(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(P2) $p(x,x) \le p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

From $(P1)$ and $(P2)$ $x = y$, if $p(x,y) = 0$, But conversely is not true. see again Matthews [15].

Notice that every partial metric on X generates a T_0 topology τ_p on X, which has as a base the family of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x)+\varepsilon\}$, for each $x \in X$ and $\varepsilon > 0$. If p is a partial metric on X, then the function $d_p : X\times X \to [0,\infty)$ define by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

which is a metric on X. And

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow p(x,x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{m,n\to+\infty} p(x_n, x_m).$$

Next, suppose that (X, p) be a partial metric space. Then,

(I) a sequences $\{x_n\} \in (X, p)$ converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;

(II) We said that a sequences $\{x_n\}$ in (X, p) is a Cauchy sequence if there exists (and is finite) $\lim_{m,n\to\infty} p(x_n, x_m)$;

(III) If every Cauchy sequence $\{x_n\} \in X$ converges with respect to τ_p to a point $x \in X$, that is $p(x, x) = \lim_{m,n\to\infty} p(x_n, x_m)$, then (X, p) is said to complete.

Next, we show the following lemmas. Which represent the connect of Cauchyness and Completeness in (X, p) and (X, d_p).

Lemma 4.2. *[15] Suppose (X, p) be a partial metric space. Let $\{x_n\}$ be any sequence in X. Then,*

(i) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p);

(ii) (X, p) is called a complete if and only if the metric space (X, d_p) is complete.

Lemma 4.3. *[16] Suppose (X, p) be a partial metric space and $\varphi : X \to [0, \infty)$ defined by $\varphi(x) = p(x, x)$. Then the function φ is lower semi-continuous in the metric space (X, d_p).*

Now, Suppose that (X, p) be a partial metric space. A and B be nonempty subset of X. By considering a non-self mapping $T : A \to B$. Then,

$$\begin{aligned} p(A, B) &:= \inf\{p(x, y) : x \in A, y \in B\}; \\ A_0 &:= \{x \in A : p(x, y) = p(A, B) \text{ for some } y \in B\}; \\ B_0 &:= \{y \in B : p(x, y) = p(A, B) \text{ for some } x \in A\}. \end{aligned}$$

Furthermore, we will see the best proximity point problem as follow.

$$\text{Find } x \in A \text{ such that } p(x, Tx) = p(A, B). \tag{4.1}$$

And let hypotheses it the followings:

$(\overline{H1})$ nonempty set A_0 is complete with respect to the topology induced by p;

$(\overline{H2})$ $T(A_0) \subseteq B_0$;

$(\overline{H3})$ there exists $\lambda : [0, \infty) \to [0, \infty)$ be given mappings such that $\lambda(t) < t$ for $t > 0, \lambda(0) = 0$ and $\limsup_{s\to t^+} \lambda(s) < t$ for all $t > 0$. Such that

$$\begin{cases} p(u, Tx) = p(A, B) \\ p(v, Ty) = p(A, B) \end{cases} \Rightarrow p(u, v) \leq \lambda(p(x, y))$$

for all $u, v, x, y \in A$;

($\overline{H4}$) there exists $\lambda : [0,\infty) \to [0,\infty)$ and $M \geq 0$ such that

$$\begin{cases} p(u,Tx) = p(A,B) \\ p(v,Ty) = p(A,B) \end{cases} \Rightarrow p(u,v) \leq \lambda(p(x,y)) + M\left(p(y,u) - \frac{p(y,y)+p(u,u)}{2}\right)$$

for all $u,v,x,y \in A$;

($\overline{H5}$) there exists $\lambda : [0,\infty) \to [0,\infty)$ and $M \geq 0$ such that

$$\begin{cases} p(u,Tx) = p(A,B) \\ p(v,Ty) = p(A,B) \end{cases} \Rightarrow p(u,v) \leq \lambda(p(x,y)) + M\left(p(x,u) - \frac{p(y,y)+p(u,u)}{2}\right)$$

for all $u,v,x,y \in A$;

Corollary 4.4. *Suppose that* $(\overline{H1}) - (\overline{H3})$ *hold, then problem (4.1) admists a unique solution.*

Proof. *Let metric* d_p *defined by* $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ $\forall x,y \in X$ *and* $\varphi : X \to [0,\infty)$ *defined by* $\varphi(x) = p(x,x)$*. Then we immediately obtain the consequence that*

$$p(x,y) = \frac{d_p(x,y) + p(x,x) + p(y,y)}{2}.$$

Therefore, the inequality $p(u,v) \leq \lambda(p(x,y))$ *leads to*

$$d_p(u,v) + \varphi(u) + \varphi(v) \leq \lambda(d_p(x,y) + \varphi(x) + \varphi(y)).$$

Next, by Theorem 3.5 and $F(a,b,c) = a+b+c,\ d = d_p$*, then we will get the following results.* □

Corollary 4.5. *Suppose that* $(\overline{H1}), (\overline{H2})$ *and* $(\overline{H4})$ *hold, then problem (4.1) admists at least a solution.*

Corollary 4.6. *Suppose that* $(\overline{H1}), (\overline{H2}), (\overline{H4})$ *and* $(\overline{H5})$ *hold, then problem (4.1) admists a unique solution.*

4.2 A variational inequality problem

Let H be a real Hilbert space, with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$. Let $K \neq \emptyset$ closed and convex subset of H and operator $S : H \to H$. Consider variational inequality problem:

$$\text{Find } u \in K \text{ such that } \langle Su, v-u \rangle \geq 0,\ \forall v \in K. \tag{4.2}$$

Suppose that $P_K : H \to K$; is a metric projection operator. Then,

$$\|u - P_K u\| \leq \|u - v\|,\ \forall v \in K \text{ holds true }.$$

Lemma 4.7. *For every* $z \in H$*,* $u \in K$ *satisfies* $\langle u-z, y-u \rangle \geq 0$*,* $\forall y \in K$ *if and only if* $u = P_K z$*.*

Lemma 4.8. *Suppose that* $S : H \to H$ *be a mapping on* H*. Then,* $u \in K$ *is a solution of* $\langle Su, v-u \rangle \geq 0$*,* $\forall v \in K$*, if and only if* $u = P_K(u - \lambda Su)$*, with* $\lambda > 0$*.*

Let $F_{ix}(S) = \{u \in H : u = Su\}$. We recommend the following important hypotheses:

(V1) $\varphi : K \to [0,\infty)$ is lower semi-continuous, with $Z_\varphi = \{u \in K : \varphi(u) = 0\}$;

(V2) $P_K(I_K - \lambda S) : K \to K$, with $\lambda > 0$, is an

(i) $(F, \lambda, \varphi)-$proximal contraction;

(ii) $(F, \lambda, \varphi)-$weak proximal contraction

Theorem 4.9. *Assume that* $(V1), (V2)(i)$ *hold, then problem (4.2) admits a unique solution* $u^* \in K$ *such that* $F_{ix}(P_K(I_K - \lambda S)) \cap Z_\varphi = \{u^*\}$. *Moreover, for each* $u_0 \in K$, *there exists a sequence* $\{u_n\} \subseteq K$ *such that* $u_{n+1} = P_K(u_n - \lambda S u_n)$ *for every* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim_{n\to\infty} u_n = u^*$.

Proof. Let operator $T : K \to K$ defined by $Tx = P_K(x - \lambda Sx)$, for all $x \in K$ if and only if $u^* = Tu^*$. Suppose that $A = B = K$ and T satisfies the hypotheses of Theorem 3.5, then we have the fixed point problem $u = Tu$ admits a unique solution $u^* \in K$. □

Next, we obtain a theorem that produce at least one solution.

Theorem 4.10. *Assume that* $(V1), (V2)(ii)$ *hold, then problem (4.2) admits at least a solution such that* $F_{ix}(P_K(I_K - \lambda S)) \cap Z_\varphi \neq \emptyset$. *Moreover, for each* $u_0 \in K$, *there exists* $\{u_n\} \subseteq K$ *such that* $u_{n+1} = P_K(u_n - \lambda S u_n)$ *for each* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim_{n\to\infty} u_n = u^* \in F_{ix}(P_K(I_K - \lambda S)) \cap Z$.

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