Coupled fixed point theorems of integral type contraction in $S_b$-metric space

D. Dhamodharan*, R. Krishnakumar1 and Stojan. Radenović2

* Department of Mathematics, Jamal Mohamed College(Autonomous), Trichy, Tamilnadu, India.
1 Department of Mathematics, Urumu Dhanlakshmi College, Trichy, Tamilnadu, India.
2 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd 35, Serbia.

Abstract: In this paper, we use integral type contraction in order to obtain some coupled fixed point theorems in the setting of $S_b$-metric space. Some results are also given in the form of corollaries.

Keywords: coupled fixed point; $S_b$ metric space; integral type contraction.

MSC: 47H10, 54H25.

1 Introduction and mathematical preliminaries

In 2012, Sedghi et al. [20] introduced the notion of $S$-metric space and proved several results, some of authors also worked on this, for example, refer [3, 4, 10, 16, 19, 21, 22, 23]. On the other hand the concept of $b$-metric space was introduced by Bakhtin [7] and Czerwik [9]. See also [8, 12, 15]

Recently Sedghi et al. [17] defined $S_b$-metric spaces by using the concepts of $S$ and $b$-metric spaces and proved common fixed point theorem for four maps in $S_b$-metric spaces. See also [2]

Bhaskar and Lakshmikantham [11] introduced the notion of coupled fixed point and they proved some coupled fixed point results also.

© by the SUMA Publishing Group.
DOI: 10.30697/rfpta-2018-032
*Corresponding author E-mail: dharan_raj28@yahoo.co.in (D Dhamodharan)
Received November 10, 2018; revised February 28, 2019; accepted March 02, 2019.
This is an open access article under the CC BY license http://creativecommons.org/licenses/by/4.0/.
The aim of this paper is to prove a unique common coupled fixed point theorem of Integral type contraction for four mappings in \( S_b \)-metric spaces. Throughout this paper \( \mathbb{R}^+ \) and \( \mathbb{N} \) denote the set of all non-negative real numbers and positive integers respectively.

First we recall some definitions, lemmas and examples.

**Definition 1.1.** [1] Let \( X \) be a nonempty set and let \( b \geq 1 \) be a given number. A function \( S_b : X^3 \rightarrow [0, \infty) \) is said to be \( S_b \)-metric if and only if for all \( x, y, z, t \in X \); the following conditions hold:

(i) \( S_b(x, y, z) = 0 \) if and only if \( x = y = z \),

(ii) \( S_b(x, x, y) = S_b(y, y, x) \) for all \( x, y \in X \),

(iii) \( S_b(x, y, z) \leq b[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)] \)

The pair \((X, S_b)\) is called an \( S_b \)-metric space.

**Remark 1.2.** [1] Note that the class of \( S_b \)-metric spaces is larger than the class of \( S \)-metric spaces. Indeed, every \( S \)-metric space is an \( S \)-metric space with \( s = 1 \). However, the converse is not always true.

**Example 1.3.** [1] Let \( X \) be a nonempty set and \( \text{card}(X) \geq 5 \). Suppose \( X = X_1 \cup X_2 \) a partition of \( X \) such that \( \text{card}(X_1) \geq 4 \). Let \( b \geq 1 \). Then

\[
S_b(x, y, z) = \begin{cases} 
0, & \text{if } x = y = z = 0; \\
3s, & \text{if } (x, y, z) \in X_1^3; \\
1, & \text{if } (x, y, z) \notin X_1^3.
\end{cases}
\]

for all \( x, y, z \in X \). \( S_b \) is a \( S_b \)-metric on \( X \) with coefficient \( b \geq 1 \).

**Definition 1.4.** [1] Let \((X, S_b)\) be an \( S_b \)-metric space and \( \{x_n\} \) be a sequence in \( X \). Then

(i) A sequence \( \{x_n\} \) is called convergent if and only if there exists \( z \in X \) such that \( S_b(x_n, x_n, z) \to 0 \) as \( n \to \infty \). In this case we write \( \lim_{n \to \infty} x_n = z \).

(ii) A sequence \( \{x_n\} \) is called Cauchy sequence if and only if \( S_b(x_n, x_n, x_m) \to 0 \) as \( n, m \to \infty \).

(iii) \((X, S_b)\) is said to be a complete \( S_b \)-metric space if every Cauchy sequence \( \{x_n\} \) converges to a point \( x \in X \) such that

\[
\lim_{n, m \to \infty} S_b(x_n, x_n, x_m) = \lim_{n \to \infty} S_b(x_n, x_n, x) = S_b(x, x, x)
\]

(iv) Define the diameter of a subset \( Y \) of \( X \) by

\[
diam(Y) := \text{Sup}\{S_b(x, y, z) | x, y, z \in X\}.
\]

Rohen et al. [2] also give the definition of \( S_b \)-metric space as follows.

**Definition 1.5.** Let \( X \) be a nonempty set and let \( b \geq 1 \) be a given number. A function \( S : X^3 \rightarrow [0, \infty) \) is said to be \( S_b \)-metric if and only if for all \( x, y, z, t \in X \); the following conditions holds:

(i) \( S(x, y, z) = 0 \) if and only if \( x = y = z \),
(ii) \( S(x, y, z) \leq b[S(x, x, t) + S(y, y, t) + S(z, z, t)] \)

the pair \((X, S)\) is called an \(S_b\)-metric space.

**Definition 1.6.** Let \( X \) be a non-empty set and \( b \geq 1 \) be a given real number. Suppose that a mapping \( S_b : X^3 \to \mathbb{R}^+ \) is a function satisfying the following properties:

\[
S_b1 \quad \int_0^{S_b(x, y, z)} \phi(t) \, dt = 0 \iff x = y = z,
\]

\[
S_b2 \quad \int_0^{S_b(x, y, z)} \phi(t) \, dt \leq b \left( \int_0^{S_b(x, x, a)} \phi(t) \, dt + \int_0^{S_b(y, y, a)} \phi(t) \, dt + \int_0^{S_b(z, z, a)} \phi(t) \, dt \right) \text{ for all } x, y, z, a \in X, \text{ where } \phi(t) \text{ is a Lebesgue integrable function which is summable non-negative and such that } \int_0^\delta \phi(t) \, dt > 0 \text{ for all } \delta > 0.
\]

Then the function \( S_b \) is called a \(S_b\)-metric on \( X \) and the pair \((X, S_b)\) is called a \(S_b\)-metric space.

**Remark 1.8.** ([17]) It should be noted that, the class of \(S_b\)-metric spaces is effectively larger than that of \(S\)-metric spaces. Indeed each \(S\)-metric space is a \(S_b\)-metric space with \(b = 1\).

Following example shows that a \(S_b\)-metric on \( X \) need not be a \(S\)-metric on \( X \).

**Example 1.9.** ([17]) Let \((X, S_b)\) be a \(S_b\)-metric space and \( S_b(x, y, z) = S^p(x, y, z), \) where \( p > 1 \) is a real number. Note that \( S_b \) is a \( \int_0^{S_b} \phi(t) \, dt \)-metric with \( b = 2^{(p-1)} \). Also, \((X, S_b)\) is not necessarily a \(S\)-metric space.

**Definition 1.10.** ([17]) Let \((X, S_b)\) be a \(S_b\)-metric space. Then, for \( x \in X, r > 0 \) we defined the open ball \( B_{S_b}(x, r) \) and closed ball \( B_{S_b}[x, r] \) with center \( x \) and radius \( r \) as follows respectively:

\[
B_{S_b}(x, r) = \{ y \in X : S_b(y, y, x) < r \},
\]

\[
B_{S_b}[x, r] = \{ y \in X : S_b(y, y, x) \leq r \}.
\]

**Lemma 1.11.** ([18]) In a \(S_b\)-metric space, we have

\[
\int_0^{S_b(x, x, y)} \phi(t) \, dt \leq b \int_0^{S_b(y, y, x)} \phi(t) \, dt
\]

and

\[
\int_0^{S_b(y, y, x)} \phi(t) \, dt \leq b \int_0^{S_b(x, x, y)} \phi(t) \, dt.
\]

**Lemma 1.12.** ([18]) In a \(S_b\)-metric space, we have

\[
\int_0^{S_b(x, x, z)} \phi(t) \, dt \leq 2b \int_0^{S_b(x, x, y)} \phi(t) \, dt + b^2 \int_0^{S_b(y, y, z)} \phi(t) \, dt.
\]

**Definition 1.13.** If \((X, S_b)\) be a \(S_b\)-metric space. A sequence \( \{x_n\} \) in \( X \) is said to be:
2.1.3 Let Theorem 2.1. D. Dhamodharan, R. Krishnakumar and Stojan. Radenović

Definition 1.14. (17) A $S_b$-metric space $(X, S_b)$ is called complete if every $S_b$-Cauchy sequence is $S_b$-convergent in $X$.

Lemma 1.15. If $(X, S_b)$ be a $S_b$-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a $S_b$-convergent to $x$, then we have

(i) \( \frac{1}{2b} \int_0^1 \phi(t) \, dt \leq \liminf_{n \to \infty} S_b(y, x_n) \)

(ii) \( \frac{1}{b} \int_0^1 \phi(t) \, dt \leq \liminf_{n \to \infty} S_b(y, x_n) \)

In particular, if $x = y$, then we have $\lim_{n \to \infty} \int_0^1 \phi(t) \, dt = 0$.

Definition 1.16. (11) Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.17. (13) Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called

(i) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = F(x, y)$ and $y = F(y, x)$,

(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = f(x) = F(x, y)$ and $y = f(y) = F(y, x)$.

Definition 1.18. (6) Let $X$ be a nonempty set and $F : X \times X \to X$ and $f : X \to X$. The $\{F, f\}$ is said to be $w$-compatible pair if $f(F(x, y)) = F(f(x), f(y))$ whenever there exist $x, y \in X$ with $f(x) = F(x, y)$ and $f(y) = F(y, x)$.

For more details of other generalized metric spaces see [14, 8, 21]

Now we give our main result.

2 Main Results

Let $\Phi$ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi$ is non-decreasing, continuous, $\phi(t) < \frac{1}{4t}$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 2.1. Let $(X, S_b)$ be a $S_b$-metric space. Suppose that $A, B : X \times X \to X$ and $P, Q : X \to X$ be satisfying

(2.1.1) $A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X)$,

(2.1.2) $\{A, P\}$ and $\{B, Q\}$ are $w$-compatible pairs,

(2.1.3) One of $P(X)$ or $Q(X)$ is $S_b$-complete subspace of $X$, 

(2.1.4) \[
2b^5 \int_0^\infty \phi(t)dt \leq \Phi \max \left\{ \begin{array}{l}
S_2\left(x,y,A(x,y),B(u,v)\right) = \int_0^\infty \phi(t)dt,
S_1\left(Px,Py,A(x,y),B(u,v)\right) = \int_0^\infty \phi(t)dt,
S_1\left(A(x,y),A(x,y),Pz\right) = \int_0^\infty \phi(t)dt,
S_2\left(A(y,x),A(y,x),Py\right) = \int_0^\infty \phi(t)dt,
S_1\left(B(u,v),B(u,v),Qu\right) = \int_0^\infty \phi(t)dt,
S_2\left(B(v,u),B(v,u),Qv\right) = \int_0^\infty \phi(t)dt,
S_2\left(A(x,y),A(x,y),Qu\right) = \int_0^\infty \phi(t)dt,
S_1\left(Px,Py,B(v,u),Qv\right) = \int_0^\infty \phi(t)dt,
S_2\left(A(y,x),A(y,x),Py\right) = \int_0^\infty \phi(t)dt,
S_1\left(B(u,v),B(u,v),Qu\right) = \int_0^\infty \phi(t)dt,
1+ \int_0^\infty \phi(t)dt
\end{array} \right\},
\]
for all \(x, y, u, v \in X, \Phi \in \Phi\), where \(\phi(t)\) is a Lebesgue integrable function which is summable nonnegative and such that \(\int_0^\infty \phi(t)dt > 0\) for all \(\delta > 0\).

Then \(A, B, P\) and \(Q\) have a unique common coupled fixed point in \(X \times X\).

**Proof.** Let \(x_0, y_0 \in X\). From (2.1.1), we can construct the sequences \(\{x_n\}, \{y_n\}, \{z_n\}\) and \(\{w_n\}\) such that
\[
A(x_{2n}, y_{2n}) = Qy_{2n+1} = z_{2n},
A(y_{2n}, x_{2n}) = Qy_{2n+1} = w_{2n},
B(x_{2n+1}, y_{2n+1}) = Px_{2n+2} = z_{2n+1},
B(y_{2n+1}, x_{2n+1}) = Px_{2n+2} = w_{2n+1}, n = 0, 1, 2, \ldots
\]

**Case (i):** Suppose \(z_{2m} = z_{2m+1}\) and \(w_{2m} = w_{2m+1}\) for some \(m\).

Assume that \(z_{2m+1} \neq z_{2m+2} \neq w_{2m+1} \neq w_{2m+2}\).

From (2.1.4), we have
\[
\int_0^\infty S_2\left(x_{2n+1}, y_{2n+1}, A(x_{2n+1}, y_{2n+1}), B(u_{2n+1}, v_{2n+1})\right)\phi(t)dt \leq 2b^5 \int_0^\infty S_3\left(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), B(u_{2n+2}, v_{2n+2})\right)\phi(t)dt
\]
\[
\leq \phi \max \left\{ \begin{array}{l}
S_b(P_{2m+2}\cdot P_{2m+2}\cdot Q_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(P_{2m+2}\cdot P_{2m+2}\cdot Q_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(A(x_{2m+2}\cdot y_{2m+2})\cdot A(x_{2m+2}\cdot y_{2m+2}), P_{2m+2})
\int_0^\varphi \phi(t)dt,

S_b(A(y_{2m+2}\cdot z_{2m+2})\cdot A(y_{2m+2}\cdot z_{2m+2}), P_{2m+2})
\int_0^\varphi \phi(t)dt,

S_b(B(x_{2m+1}\cdot y_{2m+1}), B(x_{2m+1}\cdot y_{2m+1}), Q_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(B(y_{2m+1}\cdot z_{2m+1}), B(y_{2m+1}\cdot z_{2m+1}), P_{2m+2})
\int_0^\varphi \phi(t)dt,

\end{array} \right. 
\]

\[
= \phi \max \left\{ \begin{array}{l}
S_b(z_{2m+1}\cdot z_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(w_{2m+1}\cdot w_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(z_{2m+1}\cdot z_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(w_{2m+1}\cdot w_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(x_{2m+1}\cdot x_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(w_{2m+1}\cdot w_{2m+1})
\int_0^\varphi \phi(t)dt,

S_b(x_{2m+1}\cdot x_{2m+1})
\int_0^\varphi \phi(t)dt,

\end{array} \right. 
\]
It follows that
\[
\left\{ \begin{array}{l}
\text{Case (ii):} \\
\end{array} \right.
\]
Continuing in this process we can conclude that
\[
\text{and}
\]
Thus
\[
\max \left\{ \begin{array}{l}
S_k(z_{2m+2}+z_{2m+2}+z_{2m+1}) \\
S_k(w_{2m+2}w_{2m+2}w_{2m+1}) \\
S_k(z_{2m+2}+z_{2m+2}+z_{2m+1}) \\
S_k(w_{2m+2}w_{2m+2}w_{2m+1}) \\
\end{array} \right\} \leq \phi \max \left\{ \begin{array}{l}
S_k(z_{2m+2}+z_{2m+2}+z_{2m+1}) \\
S_k(w_{2m+2}w_{2m+2}w_{2m+1}) \\
\end{array} \right\}.
\]
It follows that \(z_{2m+2} = z_{2m+1}\) and \(w_{2m+2} = w_{2m+1}\).
Continuing in this process we can conclude that \(z_{2m+k} = z_{2m}\) and
\(w_{2m+k} = w_{2m}\) for all \(k \geq 0\).

It follows that \(\{ z_m \}\) and \(\{ w_m \}\) are Cauchy sequences.

Case (ii): Assume that \(z_{2n} \neq z_{2n+1}\) or \(w_{2n} \neq w_{2n+1}\) for all \(n\).
Put\( S_n = \max \left\{ \begin{array}{l}
S_k(z_{2m+1}+z_{2m+1}) \\
S_k(w_{2m+1}w_{2m+1}w_{2m+1}) \\
\end{array} \right\} \).
From (2.1.4), we have
\[
\max \left\{ \begin{array}{l}
S_k(z_{2m+2}+z_{2m+2}+z_{2m+1}) \\
S_k(w_{2m+2}w_{2m+2}w_{2m+1}) \\
\end{array} \right\} \leq \phi \max \left\{ \begin{array}{l}
S_k(z_{2m+2}+z_{2m+2}+z_{2m+1}) \\
S_k(w_{2m+2}w_{2m+2}w_{2m+1}) \\
\end{array} \right\}.
\]
\[
\begin{align*}
\text{min} & \quad \phi \\
\leq & \quad \max \\
\left( \begin{array}{c}
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right) \\
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right) \\
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right)
\end{array} \right) \\
\leq & \quad \max \\
\left( \begin{array}{c}
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right) \\
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right) \\
\mathcal{S}_0 \left( \mathcal{Q}^2_n \right)
\end{array} \right)
\end{align*}
\]
Similarly, we can prove that
\[
\int_0^{S_{2n+1}} \varphi(t) \, dt \leq \varphi\left( \max\left\{ S_{2n+1}, S_{2n} \right\} \right).
\]
Thus
\[
\int_0^{S_{2n+1}} \varphi(t) \, dt \leq \varphi\left( \max\{ \int_0^{S_{2n}} \varphi(t) \, dt, \int_0^{S_{2n+1}} \varphi(t) \, dt \} \right).
\]
If \( S_{2n+1} \) is maximum then we get contradiction so that \( S_{2n} \) is maximum.
Thus
\[
\int_0^{S_{2n+1}} \varphi(t) \, dt \leq \varphi^\prime \left( \int_0^{S_{2n}} \varphi(t) \, dt \right) \quad (2.1)
\]
Similarly we can conclude that
\[
\int_0^{S_{2n}} \varphi(t) \, dt < \int_0^{S_{2n-1}} \varphi(t) \, dt.
\]
It is clear that \( \{S_n\} \) is a non-increasing sequence of non-negative real numbers and must converge to a real number say \( r \geq 0 \).
Suppose \( r > 0 \).
Letting \( n \to \infty \), in (2.1), we have \( r \leq \varphi(r) < r \).
It is contradiction. Hence \( r = 0 \)
Thus
\[
\lim_{n \to \infty} \int_0^{S_{2n+1}} \varphi(t) \, dt = 0 \quad (2.2)
\]
and
\[
\lim_{n \to \infty} \int_0^{S_{2n}} \varphi(t) \, dt = 0. \quad (2.3)
\]
Now we prove that \( \{z_{2n}\} \) and \( \{w_{2n}\} \) are Cauchy sequence in \((X, S_b)\). On contrary we suppose that \( \{z_{2n}\} \) or \( \{w_{2n}\} \) is not Cauchy. Then there exist \( \epsilon > 0 \) and monotonically increasing sequence of natural numbers \( \{2m_k\} \) and \( \{2n_k\} \) such that
\[
\max\{ \int_0^{z_{2m_k}} \varphi(t) \, dt, \int_0^{z_{2n_k}} \varphi(t) \, dt \} \geq \epsilon \quad (2.4)
\]
and
\[
\max\{ \int_0^{w_{2m_k}} \varphi(t) \, dt, \int_0^{w_{2n_k}} \varphi(t) \, dt \} < \epsilon. \quad (2.5)
\]
From (2.4) and (2.5), we have
\[
\epsilon \leq \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4}}}})} \varphi(t) \, dt \right\}
\]

\[
\leq 2b \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4}}}})} \varphi(t) \, dt \right\}
\]

\[
+ b \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\}
\]

\[
\leq 2b \left( 2b \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\} \right)
\]

\[
+ b \left( b \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\} \right)
\]

\[
+ \left( b \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\} \right)
\]

\[
(2.6)
\]

\[
\leq 4b^3 \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\}
\]

\[
+ 2b^2 \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\}
\]

\[
+ 2b^3 \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\}
\]

\[
+ b^2 \max\left\{ \int_0^{S_k(z_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt, \int_0^{S_k(w_{2m_1z_{2m_2z_{2m_3z_{2m_4+1}}}})} \varphi(t) \, dt \right\}.
\]

(2.7)

Now from (2.1.4), we have
\[\begin{align*}
2\beta^5 \int_0^\infty \varphi(t) dt & \leq \phi \max \left\{ \begin{array}{ll}
S_b(2^n_{2n+2} \varphi_{2n+2} + 2) + \phi(t) dt, \\
S_b(2^n_{2n+2} \varphi_{2n+2} + 1) + \phi(t) dt, \\
S_b(2^n_{2n+2} \varphi_{2n+2} + 1) + \phi(t) dt,
\end{array} \right. \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
\[
\begin{aligned}
&\leq \phi \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad \leq 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad \leq 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + b^2 \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad \leq 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + b^2 \left( 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \right) \\
&\quad < 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + 2b^3 + b^3 \left( 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \right) \\
&\quad \leq 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + b^3 \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad \leq 2b \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + 2b^3 + 2b^4 \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\} \\
&\quad + b^5 \max \left\{ \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt, \int_0^{\varphi(t)} dt \right\}.
\end{aligned}
\]
Letting $k \to \infty$, we have
\[
\lim_{k \to \infty} \max \left\{ \int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt, \int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt \right\} \leq 2b^3e.
\]
Also
\[
\lim_{k \to \infty} \frac{S_k(2w_{2m_k+1-2m_k+1-2m_k})}{\int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt} \leq 1 + \int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt.
\]
\[
\leq \lim_{k \to \infty} \frac{S_k(2w_{2m_k+1-2m_k+1-2m_k})}{\int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt} \leq 2b^3e.
\]

Similarly
\[
\lim_{k \to \infty} \max \left\{ \int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt, \int_0^{2w_{2m_k+1-2m_k+1-2m_k}} \varphi(t) dt \right\} \leq 2b^3e.
\]
Letting $k \to \infty$ in (2.8), we have
\[
\lim_{k \to \infty} \max \left\{ \int_0^{2w_{2m_k+1-2m_k+1-2m_k+1}} \varphi(t) dt, \int_0^{2w_{2m_k+1-2m_k+1-2m_k+1}} \varphi(t) dt \right\} \leq \frac{1}{2b^3} \varphi \left( \max \{2b^3e, 0, 0, 0, 2b^6e, 2b^6e \} \right)
\]
\[
= \frac{1}{2b^3} \varphi(2b^6e).
\]

Now letting $n \to \infty$ in (2.7), from (2.2), (2.3) and (2.9), we have
\[
e \leq 0 + 0 + 0 + b^2 \frac{1}{2b^3} \varphi(2b^6e) < e.
\]

It is a contradiction.

Hence \{z_{2n}\} and \{w_{2n}\} are $S_k$-Cauchy sequences in $(X, S_k)$.
In addition
\[
\max \left\{ \int_0^{S_b(z_{2n+1}, z_{2m+1})} \varphi(t) dt, \int_0^{S_b(w_{2n+1}, w_{2m+1})} \varphi(t) dt \right\}
\leq 2b \max \left\{ \int_0^{S_b(z_{2n+1}, 2z_{2n+1})} \varphi(t) dt, \int_0^{S_b(w_{2n+1}, 2w_{2n+1})} \varphi(t) dt \right\}
\]
\[
+ b \max \left\{ \int_0^{S_b(z_{2n+1} + z_{2n+1}, 2z_{2n+1})} \varphi(t) dt, \int_0^{S_b(w_{2n+1} + w_{2n+1}, 2w_{2n+1})} \varphi(t) dt \right\}
\]
\[
+ 2b^2 \max \left\{ \int_0^{S_b(z_{2n+1}, z_{2n+1} + 2z_{2n+1})} \varphi(t) dt, \int_0^{S_b(w_{2n+1}, w_{2n+1} + 2w_{2n+1})} \varphi(t) dt \right\}
\]
\[
+ b^2 \max \left\{ \int_0^{S_b(z_{2n+1}, z_{2n+1} + z_{2n+1})} \varphi(t) dt, \int_0^{S_b(w_{2n+1}, w_{2n+1} + w_{2n+1})} \varphi(t) dt \right\}.
\]

From (2.2), (2.3) and since \( \{z_{2n}\} \) and \( \{w_{2n}\} \) are \( S_b \)-Cauchy sequences, it follows that \( \{z_{2n+1}\} \) and \( \{w_{2n+1}\} \) are also \( S_b \)-Cauchy sequences in \( (X, S_b) \).

Thus \( \{z_n\} \) and \( \{w_n\} \) are \( S_b \)-Cauchy sequences in \( (X, S_b) \).

Suppose \( P(X) \) is \( S_b \)-complete subspace of \( (X, S_b) \).

Then the sequences \( \{z_n\} \) and \( \{w_n\} \) converge to \( a \) and \( \beta \) respectively in \( P(X) \).

Thus there exist \( a \) and \( b \) in \( P(X) \) such that
\[
\lim_{n \to \infty} z_n = a = Pa \quad \text{and} \quad \lim_{n \to \infty} w_n = \beta = Pb.
\]
(2.10)

Now we have to prove that \( A(a, b) = a \) and \( A(b, a) = \beta \).

From (2.14) and Lemma (1.11), we have
\[
\frac{1}{2b} \int_0^{S_b(A(a, b), A(a, b), b)} \varphi(t) dt \leq \lim_{n \to \infty} \inf 2b^5 \int_0^{S_b(A(a, b), A(a, b), b(S_b(z_{2n+1}, w_{2n+1})))} \varphi(t) dt
\]
$$\leq \liminf_{n \to \infty} \phi \max \left\{ \begin{array}{c} S_h(P_a,P_b,Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(A(a,b),A(a,b),P_b) \int_0^\infty \varphi(t)dt, \\
S_h(B(y_{2n+1},y_{2n+1}),B(y_{2n+1},y_{2n+1}),Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(B(y_{2n+1},y_{2n+1}),B(y_{2n+1},y_{2n+1}),Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(A(a,b),A(a,b),Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(B(y_{2n+1},y_{2n+1}),B(y_{2n+1},y_{2n+1}),P_b) \int_0^\infty \varphi(t)dt, \\
S_h(P_b,P_b,Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt \end{array} \right\}$$

$$\leq \liminf_{n \to \infty} \phi \max \left\{ \begin{array}{c} S_h(A(a,b),A(\alpha,\beta),Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(A(\alpha,\beta),A(\alpha,\beta),\alpha) \int_0^\infty \varphi(t)dt, \\
S_h(S_{t_{2n+1}},S_{t_{2n+1}},Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(S_{t_{2n+1}},S_{t_{2n+1}},Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(S_{t_{2n+1}},S_{t_{2n+1}},Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt, \\
S_h(A(\alpha,\beta),A(\alpha,\beta),Q_{t_{2n+1}}) \int_0^\infty \varphi(t)dt \end{array} \right\}$$
\[
\phi \left( \max \left\{ \begin{array}{l}
S_b(A(\alpha, \beta), A(\alpha, \beta), \alpha) \\
 \int_0^\infty \phi(t)dt, \\
S_b(A(\beta, \alpha), A(\beta, \alpha), \beta)
\end{array} \right\}, 0, 0, 0, 0 \right)
\]

Similarly
\[
\frac{1}{2b} \int_0^\infty \phi(t)dt \leq \phi \left( \max \left\{ \begin{array}{l}
S_b(A(\alpha, \beta), A(\alpha, \beta), \alpha) \\
 \int_0^\infty \phi(t)dt, \\
S_b(A(\beta, \alpha), A(\beta, \alpha), \beta)
\end{array} \right\} \right).
\]

Thus
\[
\frac{1}{2b} \max \left\{ \begin{array}{l}
S_b(A(\alpha, \beta), A(\alpha, \beta), \alpha) \\
 \int_0^\infty \phi(t)dt, \\
S_b(A(\beta, \alpha), A(\beta, \alpha), \beta)
\end{array} \right\} \leq \phi \left( \max \left\{ \begin{array}{l}
S_b(A(\alpha, \beta), A(\alpha, \beta), \alpha) \\
 \int_0^\infty \phi(t)dt, \\
S_b(A(\beta, \alpha), A(\beta, \alpha), \beta)
\end{array} \right\} \right).
\]

By the definition of \(\phi\), it follows that \(A(a, b) = \alpha = Pa\) and \(A(b, a) = \beta = Pb\). Since \((A, P)\) is \(w\)-compatible pair, we have \(A(\alpha, \beta) = Pa\) and \(A(\beta, \alpha) = Pb\). From (2.1.4) and Lemma (1.11), we have
\[
\frac{1}{2b} \int_0^\infty \phi(t)dt \leq \lim_{n \to \infty} \sup 2^5 \int_0^\infty \phi(t)dt.
\]
\[
\begin{aligned}
\leq \lim_{n \to \infty} \sup_{\phi} \max \left\{ \begin{array}{l}
S_p(P_n,P_n,Q_{2n+1}) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(A(\alpha,\beta),A(\alpha,\beta),P_n) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(A(\beta,\alpha),A(\beta,\alpha),P_n) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(B(2n+1,2n+2),B(2n+1,2n+2),Q_{2n+1}) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(B(2n+1,2n+2),B(2n+1,2n+2),P_n) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(P_n,P_n,Q_{2n+1}) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(A(\alpha,\beta),A(\alpha,\beta),Q_{2n+1}) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(B(2n+1,2n+2),B(2n+1,2n+2),P_n) \int_0^{\phi(t)dt} \phi(t)dt, \\
S_p(0,0) \int_0^{\phi(t)dt} \phi(t)dt
\end{array} \right\},
\end{aligned}
\]
Thus

\[
\lim_{n \to \infty} \sup \phi \max \left\{ \begin{array}{l}
S_3(A(a,\beta),A(a,\beta),z_{2n}) \phi(t)dt, \\
S_5(A(\alpha,\beta),A(\beta,\alpha),\alpha_{2n}) \phi(t)dt,
\end{array} \right\}
\]

Similarly

\[
\frac{1}{2b} \int_0^{2b} \phi(t)dt \leq \phi \left( 2b^2 \max \left\{ \begin{array}{l}
S_3(A(a,\beta),A(a,\beta),\alpha) \phi(t)dt, \\
S_5(A(\alpha,\beta),A(\beta,\alpha),\alpha) \phi(t)dt,
\end{array} \right\} \right).
\]

Thus

\[
\frac{1}{2b} \max \left\{ \begin{array}{l}
S_3(A(a,\beta),A(a,\beta),\alpha) \phi(t)dt, \\
S_5(A(\alpha,\beta),A(\beta,\alpha),\alpha) \phi(t)dt,
\end{array} \right\} \leq \phi \left( 2b^2 \max \left\{ \begin{array}{l}
S_3(A(a,\beta),A(a,\beta),\alpha) \phi(t)dt, \\
S_5(A(\alpha,\beta),A(\beta,\alpha),\alpha) \phi(t)dt,
\end{array} \right\} \right).
\]

By the definition of \( \phi \), it follows that \( A(\alpha, \beta) = a = Pa \) and \( A(\beta, a) = \beta = P\beta \).

Therefore \( (a, \beta) \) is common coupled fixed point of \( A \) and \( P \).

Since \( A(X \times X) \subseteq Q(X) \) there exist \( x \) and \( y \) in \( X \) such that \( A(a, \beta) = a = Qx \) and \( A(\beta, a) = \beta = Qy \).
From (2.1.4), we have
\[
\int_0^1 \varphi(t) \, dt = \int_0^1 \varphi(t) \, dt
\]
\[
\leq 2 \beta S_\beta \left( \int_0^1 \varphi(t) \, dt \right)
\]
\[
\leq \phi \left\{ \max \left\{ \begin{array}{c}
S_\beta (B(x,y), B(x,y), A) \\
S_\beta (B(y,x), B(y,x), A) \\
S_\beta (B(x,y), B(y,x), A)
\end{array} \right\} \right\}
\]
\[
= \phi \left\{ \max \left\{ \begin{array}{c}
S_\beta (B(x,y), B(x,y), A) \\
S_\beta (B(y,x), B(y,x), A) \\
S_\beta (B(x,y), B(y,x), A)
\end{array} \right\} \right\}
\]
\[
\leq \phi \left\{ b \max \left\{ \begin{array}{c}
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y))
\end{array} \right\} \right\}
\]
Similarly
\[
\int_0^1 \varphi(t) \, dt \leq \phi \left\{ \max \left\{ \begin{array}{c}
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y))
\end{array} \right\} \right\}
\]
Thus
\[
\max \left\{ \begin{array}{c}
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(y,x))
\end{array} \right\}
\]
\[
\leq \phi \left\{ b \max \left\{ \begin{array}{c}
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y)) \\
S_\beta (a, a, B(x,y))
\end{array} \right\} \right\}
\]
It follows that \( B(x, y) = a = Qx \) and \( B(y, x) = b = Qy \). Since \((B, Q)\) is \(\omega\)-compatible pair, we have \(B(a, b) = Qa\) and \(B(b, a) = Qb\).

From (2.1.4) we have
\[
\int_0^1 \varphi(t) \, dt = \int_0^1 \varphi(t) \, dt
\]
To prove uniqueness let us take $B(\alpha,\beta) = \alpha = Q\alpha$ and $B(\beta,\alpha) = \beta = Q\beta$.

Therefore $(\alpha,\beta)$ is common coupled fixed point of $A, B, P$ and $Q$.

To prove uniqueness let us take $S^1, B^1$ is another common coupled fixed point of $A, B, P$ and $Q$. From (2.14) we have

$$\int_0^1 \phi'(t) dt \leq 2 b^5 S_N(A(\alpha,\beta),A(\alpha,\beta),B(\alpha,\beta))$$

$\leq \phi \max \left\{ \begin{array}{l} S_N(P_\alpha P_\beta Q_\alpha) S_N(P_\beta P_\alpha Q_\beta) \\ S_N(A(\alpha,\beta),A(\beta,\alpha),A(\beta,\alpha)) S_N(A(\beta,\alpha),A(\beta,\alpha),P_\beta) \\ S_N(B(\alpha,\beta),B(\alpha,\beta),Q_\alpha) S_N(B(\beta,\alpha),B(\beta,\alpha),Q_\beta) \\ S_N(A(\alpha,\beta),A(\alpha,\beta),Q_\alpha) S_N(B(\beta,\alpha),B(\beta,\alpha),P_\beta) \\ S_N(P_\alpha P_\beta P_\alpha Q_\beta) S_N(P_\beta P_\alpha P_\beta Q_\beta) \\ S_N(P_\alpha P_\beta P_\alpha) S_N(P_\beta P_\alpha P_\beta) \\ \end{array} \right\}.$

Similarly

$$\int_0^1 \phi'(t) dt \leq \phi \max \left\{ \begin{array}{l} S_N(A(\alpha,\beta),B(\alpha,\beta)) S_N(B(\beta,\alpha),B(\beta,\alpha)) \\ S_N(B(\alpha,\beta),B(\alpha,\beta),\alpha) S_N(B(\beta,\alpha),B(\beta,\alpha),\beta) \\ S_N(A(\alpha,\beta),B(\alpha,\beta)) S_N(B(\beta,\alpha),B(\beta,\alpha)) \\ \end{array} \right\}.$$

Thus

$$\max \left\{ \begin{array}{l} S_N(A(\alpha,\beta),B(\alpha,\beta)) \\ S_N(B(\beta,\alpha),B(\beta,\alpha)) \end{array} \right\} \leq \phi \max \left\{ \begin{array}{l} S_N(A(\alpha,\beta),B(\alpha,\beta)) \\ S_N(B(\beta,\alpha),B(\beta,\alpha)) \end{array} \right\}.$$

It follows that $B(\alpha,\beta) = \alpha = Q\alpha$ and $B(\beta,\alpha) = \beta = Q\beta$.

Therefore $(\alpha,\beta)$ is common coupled fixed point of $A, B, P$ and $Q$.

To prove uniqueness let us take $(\alpha^1, \beta^1)$ is another common coupled fixed point of $A, B, P$ and $Q$. From (2.14) we have
Let \( X \) be a metric space with \( b \). Let \( \alpha \) and \( \beta \) be\( \in [0,1] \) such that \( \alpha \neq \beta \), then \( S_b(x,y, z) = \left( |y + z - 2x| + |y - z| \right)^2 \). Define \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( \phi(t) = \frac{t}{4} \) also \( A, B : X \times X \times X \rightarrow \mathbb{R}^+ \) and \( P, Q : X \rightarrow \mathbb{R}^+ \) by \( A(x, y) = \frac{x^2 + y^2}{8} \), \( B(x, y) = \frac{x^2 + y^2}{9} \), \( P(x) = \frac{4}{5} \) and \( Q(x) = \frac{2}{10} \). Define \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( \phi(t) = t \). Let \( x, y, z \in X \) and \( x \neq y \), then

\[
S_b(A(x, y), A(x, y), B(u, v)) \leq S_b(x, y, z) \leq S_b(A(x, y), A(x, y), B(u, v))
\]

From (2.2), we have

\[
S_b(a, a, a) \leq 2 b^5 S_b(A(x, y), A(x, y), B(u, v))
\]

\[
= 2 b^5 \left( \int_0^1 \phi(t) dt \right)
\]

\[
\leq 2 b^5 \left( \max \left\{ \int_0^1 \phi(t) dt, \int_0^1 \phi(t) dt \right\} \right)
\]

Similarly,

\[
S_b(\beta, \beta, \beta) \leq \phi(\max \left\{ \int_0^1 \phi(t) dt, \int_0^1 \phi(t) dt \right\} )
\]

Thus

\[
\max \left\{ \int_0^1 \phi(t) dt, \int_0^1 \phi(t) dt \right\} \leq \phi \left( \max \left\{ S_b(a, a, a), S_b(\beta, \beta, \beta) \right\} \right)
\]

It follows that \( a = a_1 \) and \( \beta = \beta_1 \). Hence \((a, \beta)\) is unique common coupled fixed point of \( A, B, P \) and \( Q \).\\

**Example 2.2.** Let \( X = [0, 1] \) and \( S_b : X \times X \rightarrow \mathbb{R}^+ \) by \( S_b(x, y, z) = \left( |y + z - 2x| + |y - z| \right)^2 \), then \( S_b \) is \( S_b \)-metric space with \( b = 4 \). Define \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( \phi(t) = \frac{t}{4} \) also \( A, B : X \times X \rightarrow \mathbb{R}^+ \) by \( A(x, y) = \frac{x^2 + y^2}{8} \), \( B(x, y) = \frac{x^2 + y^2}{9} \), \( P(x) = \frac{4}{5} \) and \( Q(x) = \frac{2}{10} \). Define \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( \phi(t) = t \). Let \( x, y, z \in X \) and \( x \neq y \), then

\[
2b^5 \int_0^1 \phi(t) dt \leq \frac{1}{2(4^2)} \max \left\{ S_b(Px, Px, Qy), S_b(Py, Py, Qz) \right\}
\]
It is clear that all conditions of Theorem 2.1 are satisfy and \((0,0)\) is unique common coupled fixed point of \(A, B, P\) and \(Q\).

**Theorem 2.3.** Let \((X, S_b)\) be a complete \(S_b\)-metric space. Suppose that \(A : X \times X \rightarrow X\) be satisfying
Corollary 2.4. Let \((X, S_\delta)\) be a \(S_\delta\)-metric space. Suppose that \(A, B : X \times X \to X\) and \(P, Q : X \to X\) be satisfying

\begin{enumerate}
  \item \(A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X),\)
  \item \(\{A, P\}\) and \(\{B, Q\}\) are \(w\)-compatible pairs,
  \item One of \(P(X)\) or \(Q(X)\) is \(S_\delta\)-complete subspace of \(X,\)
  \item \(S_\delta(A(x, y), A(x, y), B(u, v))\)
\end{enumerate}

for all \(x, y, u, v \in X, \phi \in \Phi\), where \(\phi(t)\) is a Lebesgue integrable function which is summable nonnegative and such that \(\int_0^\delta \phi(t)dt > 0\) for all \(\delta > 0\).

Then \(A, B, P\) and \(Q\) have a unique common coupled fixed point in \(X \times X\).

Corollary 2.5. Let \((X, S_\delta)\) be a complete \(S_\delta\)-metric space. Suppose that \(A : X \times X \to X\) be satisfying

\begin{enumerate}
  \item \(A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X),\)
  \item \(\{A, P\}\) and \(\{B, Q\}\) are \(w\)-compatible pairs,
  \item One of \(P(X)\) or \(Q(X)\) is \(S_\delta\)-complete subspace of \(X,\)
  \item \(S_\delta(A(x, y), A(x, y), B(u, v))\)
\end{enumerate}

for all \(x, y, u, v \in X, \phi \in \Phi\), where \(\phi(t)\) is a Lebesgue integrable function which is summable nonnegative and such that \(\int_0^\delta \phi(t)dt > 0\) for all \(\delta > 0\).

Then \(A, B, P\) and \(Q\) have a unique common coupled fixed point in \(X \times X\).
\[
\begin{align*}
\phi(t) := & \int_{0}^{t} \psi(s) \, ds, \\
\text{for all } x, y, u, v \in \Phi, \text{ where } \phi(t) \text{ is a Lebesgue integrable function which is summable nonnegative and such that } \int_{0}^{\infty} \phi(t) \, dt > 0 \text{ for all } \delta > 0. \\
\text{Then } A \text{ have a unique coupled fixed point in } X \times X.
\end{align*}
\]
D. Dhamodharan, R. Krishnakumar and Stojan. Radenović