Abstract: The aim of this paper is to present some common fixed point results for four weakly compatible mappings in complex valued $S$-metric spaces. Also, new contraction conditions in this space are given. Finally, as applications of our results, we give the existence and uniqueness of common solution of Hammerstein and Urysohn integral equations.

Keywords: Complex valued $S$-metric spaces; common fixed point; nonlinear integral equations.

MSC: 47H10, 55H02.

1 Introduction

Fixed point theory is very important and useful theory in various branches as determining the existence and uniqueness of solutions to many mathematical equations in mathematical science, engineering and applications in other fields. The existence and uniqueness of fixed points in different metric spaces is very famous problem. Banach's contraction principle play an important role as the most widely used fixed point theorem in all analysis.
In 1997, Popa [13] gave the definition of an implicit relation, which is cover several well known contractions of the existing literature. So, many authors showed several fixed point results under this concept (see [11,14,15,16]). In fact, the force of implicit relations lies in their unifying power besides being general enough to a multitude yield new contraction.

In 2011, Azam et al. [3] introduced the concept of complex valued metric space which is more general of the classical metric space and established some fixed point results for mappings involving rational expressions. Subsequently, many authors have studied the existence and uniqueness of the fixed points and common fixed points of self mapping in complex valued metric spaces (see[2,6-10,17-23]).

In 2014, Mlaiki [12] gave the concept of complex valued S-metric spaces as generalization of complex valued metric spaces and obtained some common fixed point results in this space.

According to this line, we use the idea of implicit relation in complex valued S-metric spaces to prove the common fixed point theorems for four weakly compatible mappings. Next, we give a new contraction conditions as a corollary. Finally, we strength our results by giving some applications to find analytical solution for nonlinear integral equations.

2 Preliminaries

In this section, we recall some notations and definitions due to Azam et al. [3] and Mlaiki [12], that will be used in prove our results.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ iff } \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

(C1) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2),$

(C2) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2),$

(C3) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2),$

(C4) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2).$

In particular, we write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (C1), (C2) and (C3) is satisfied and we write $z_1 \prec z_2$ if only (C3) is satisfied.

Definition 2.1 [3] Let $X$ be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued metric on $X$ if the following conditions are satisfied:

(M1) $0 \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \iff x = y,$

(M2) $d(x,y) = d(y,x)$ for all $x, y \in X,$

(M3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X.$

In this case, we say that $(X,d)$ is called a complex valued metric space.
Example 2.1 [5] Let $X = C$. Define a mapping $d : C \times C \to C$ by
\[ d(z_1, z_2) = d^k |z_1 - z_2| \quad \forall \ z_1, z_2 \in C, \]
where $k \in [0, \pi/2]$. Then $(X, d)$ is called a complex valued metric space.

Definition 2.2 [4] Let $X$ be a non-empty set and $(S, T)$ be a pair of self-mappings on $X$. Then $(S, T)$ is said to be weakly compatible if
\[ Sx = Tx \Rightarrow STx = TSx \quad \forall \ x \in X. \]

Definition 2.3 [3] Let $\{x_r\}$ be a sequence in a complex valued metric space $(X, d)$ and $x \in X$. Then
(i) $x$ is called the limit of $\{x_r\}$ if for every $\epsilon > 0$ there exist $r_0 \in N$ such that $d(x_r, x) < \epsilon$ for all $r > r_0$ and we can write $\lim_{r \to \infty} x_r = x$.
(ii) $\{x_r\}$ is called a Cauchy sequence if for every $\epsilon > 0$ there exist $r_0 \in N$ such that $d(x_r, x_{r+s}) < \epsilon$ for all $r > r_0$, where $s \in N$.
(iii) $(X, d)$ is said to be a complete complex valued metric space if every Cauchy sequence is convergent in $(X, d)$.

Lemma 2.1 [3] Let $(X, d)$ be a complex valued metric space and $\{x_r\}$ be a sequence in $X$. Then $\{x_r\}$ converges to $x$ if and only if $|d(x_r, x)| \to 0$ as $r \to \infty$.

Lemma 2.2 [3] Let $(X, d)$ be a complex valued metric space. Then a sequence $\{x_r\}$ in $X$ is a Cauchy sequence if and only if $|d(x_r, x_{r+s})| \to 0$ as $r \to \infty$, where $s \in N$.

Definition 2.4 [12] Let $X$ be a nonempty set and $C$ be the set of complex numbers. A function $S : X^3 \to C$ is called a complex valued $S$-metric on $X$ if the following conditions are satisfied:

(S1) $0 \preceq S(x, y, z),$
(S2) $S(x, y, z) = 0 \iff x = y = z,$
(S3) $S(x, y, z) \preceq S(x, x, t) + S(y, y, t) + S(z, z, t).$

for all $x, y, z, t \in X$. In this case, we say that $(X, S)$ is called a complex valued $S$-metric space.

Example 2.2 [12] Let $X = C$. Define a mapping $S : C^3 \to C$ by
\[ S(z_1, z_2, z_3) = \max \{Re(z_1), Re(z_2)\} - Re(z_2) + i \max \{Im(z_1), Im(z_2)\} - Im(z_2). \]
Then $(C, S)$ is called a complex valued $S$-metric space.

Definition 2.5 [12] Let $(X, S)$ be a complex valued $S$-metric space, then
(i) A sequence $\{x_r\}$ in $X$ converges to $x$ if and only if for every $0 < \epsilon \in C$, there exist $r_0 \in N$ such that $S(x_r, x, x) < \epsilon$ for all $r \geq r_0$ and we can write $\lim_{r \to \infty} x_r = x$. 

(ii) A sequence \( \{x_r\} \) in \( X \) is called a Cauchy sequence if for every \( 0 < \epsilon \in C \), there exist \( r_0 \in \mathbb{N} \) such that \( S(x_r, x_s, x_s) < \epsilon \) for all \( r, s \geq r_0 \), where \( s \in \mathbb{N} \).

(iii) An S-metric space \((X, S)\) is said to be a complete if every Cauchy sequence is convergent.

**Lemma 2.3** [12] Let \((X, S)\) be a complex valued S-metric space and \( \{x_r\} \) be a sequence in \( X \). Then \( \{x_r\} \) converges to \( x \) if and only if and only if \( |S(x_r, x_r, x)| \to 0 \) as \( r \to \infty \).

**Lemma 2.4** [12] Let \((X, S)\) be a complex valued S-metric space and \( \{x_r\} \) be a sequence in \( X \). Then \( \{x_r\} \) is a Cauchy sequence if and only if \( |S(x_r, x_r, x_{r+s})| \to 0 \) as \( r \to \infty \), where \( s \in \mathbb{N} \).

**Lemma 2.5** [12] If \((X, S)\) be a complex valued S-metric space, then

\[
S(x, x, y) = S(y, x, x) \quad \text{for all } x, y \in X.
\]

**Definition 2.6** [1] The required control functions are defined as follows:

(i) \( \psi : C^+ \to C^+ \) is a continuous nondecreasing function with \( \psi(z) = 0 \) if and only if \( z = 0 \),

(ii) \( \phi : C^+ \to C^+ \) is lower semi-continuous function with \( \phi(z) = 0 \) if and only if \( z = 0 \).

Respectively, we denote the set of all \( \psi \)'s and the set of all \( \phi \)'s by \( \Psi \) and \( \Phi \).

3 Main Results

We begin with the following definition:

**An Implicit Relation.** Let \( f \) be the set of all complex valued lower semi-continuous functions \( \alpha : C^6 \to C \) satisfying the following conditions:

\( a_1 \) \( \alpha \) is non-increasing in the 5th variable,

\( a_2 \) for \( u, v \geq 0 \), there exists \( q \in [0, 1) \) such that \( |u| \leq q |v| \) if \( \alpha(u, v, v, u, u + 2v, 0) \leq 0 \),

\( a_3 \) \( \alpha(u, 0, 0, u, u) > 0 \) or \( \alpha(u, 0, 0, u, 0) > 0 \) for all \( u > 0 \).

**Remark 3.1.** All examples 2.2-2.22 in [1] are satisfied with our implicit relation by adding simple changes. These changes appearing in Corollary 3.1.

Now, we show some common fixed point theorems in complex valued \( S \)-metric space.

**Theorem 3.1.** Let \( f, g, P \) and \( Q \) be four self-mappings on a complete complex valued \( S \)-metric space \((X, S)\) such that \( f(X) \subseteq P(X) \) and \( g(X) \subseteq Q(X) \). Assume that there exists \( \alpha \in F \) such that for all \( x, y \in X, x \neq y \),

\[
\alpha(S(fx, fx, gy), S(Qx, Qx, Py), S(Qx, Qx, fx), S(Py, Py, gy)) \geq 0 \quad \text{and} \quad S(gy, gy, Qx), S(fx, fx, Py)) \leq 0.
\] (1)
If \( f(X) \cup g(X) \) is complete subspace of \( X \), then the pairs \( (f, Q) \) and \( (g, P) \) have a unique common point of coincidence. Moreover, if the pairs \( (f, Q) \) and \( (g, P) \) are weakly compatible, then the four mappings have a unique common fixed point.

**Proof.** Let \( x_0 \) be arbitrary point in \( X \). Since \( f(X) \subseteq P(X) \) and \( g(X) \subseteq Q(X) \), then we can define two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that,

\[
\begin{align*}
y_{2n+1} &= Px_{2n+1} = fx_{2n}, \\
y_{2n+2} &= Qx_{2n+2} = gx_{2n+1}.
\end{align*}
\]

Since \( \{y_n\} \subseteq f(X) \cup g(X) \). Now, we show that \( \{y_n\} \) is a Cauchy sequence. Taking \( x = x_{2n} \) and \( y = x_{2n+1} \) in (1), we get

\[
a \{S(fx_{2n}, fx_{2n}, gx_{2n+1}), S(Qx_{2n}, Qx_{2n}, Px_{2n+1})
\]

\[
\cdot S(Qx_{2n}, Qx_{2n}, fx_{2n}), S(Px_{2n+1}, Px_{2n+1}, gx_{2n+1})
\]

\[
\cdot S(gx_{2n+1}, gx_{2n+1}, Qx_{2n}), S(fx_{2n}, fx_{2n}, Px_{2n+1}) \}
\]

\( \sum_{i=0}^{n} a_{i} \leq 0. \)

This implies that

\[
a \{S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n}, y_{2n}, y_{2n+1})
\]

\[
\cdot S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n+2}, y_{2n+2}, y_{2n}) \}

that is,

\[
a \{S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n}, y_{2n}, y_{2n+1})
\]

\[
\cdot S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n}, y_{2n}, y_{2n+2}), 0 \}
\]

From \( a_1 \) and triangle inequality, we get

\[
a \{S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n}, y_{2n}, y_{2n+1})
\]

\[
\cdot S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n+1}, y_{2n+1}, y_{2n+2})
\]

\[
\cdot 2S(y_{2n}, y_{2n}, y_{2n+1}) + S(y_{2n+2}, y_{2n+2}, y_{2n+1}), 0 \}
\]

that is,

\[
a \{S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n}, y_{2n}, y_{2n+1})
\]

\[
\cdot S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n+1}, y_{2n+1}, y_{2n+2})
\]

\[
\cdot 2S(y_{2n}, y_{2n}, y_{2n+1}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2}), 0 \}
\]

From \( a_2 \), we have
\[ |S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq q |S(y_{2n}, y_{2n}, y_{2n+1})|. \]

By a similar way, one can write
\[ |S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leq q |S(y_{2n+1}, y_{2n+1}, y_{2n+2})|. \]

Consequently,
\[ |S(y_{n+1}, y_{n+1}, y_{n+2})| \leq q |S(y_n, y_n, y_{n+1})| \leq \ldots \leq q^{n+1} |S(y_0, y_0, y_1)|. \]

Also, for any \( n > m, \) we get
\[
|S(y_n, y_m, y_n)| \leq 2 \left( |S(y_m, y_m, y_{m+1})| + |S(y_{m+1}, y_{m+1}, y_{m+2})| + \ldots + |S(y_n-1, y_n-1, y_n)| \right)
\]
\[
\leq 2 \left( q^m + q^{m+1} + \ldots + q^{n-1} \right) |S(y_0, y_0, y_1)|
\]
\[
\leq 2 \frac{q^m}{1 - q} |S(y_0, y_0, y_1)| \to 0, \text{ as } m, n \to \infty.
\]

This shows that \( \{y_n\} \) is a Cauchy sequence in \( X. \) Since \( (X, S) \) is complete, there exists \( u \in X \) such that \( y_n \to u \) as \( n \to \infty. \) Then from (2), we obtain
\[
\lim_{n \to \infty} f x_{2n} = \lim_{n \to \infty} P x_{2n+1} = \lim_{n \to \infty} Q x_{2n} = \lim_{n \to \infty} g x_{2n+1} = u. \tag{3}
\]

Since \( f(X) \subseteq P(X), \) if \( u \in g(X), \) there exists \( v \in X \) such that
\[
P v = u. \tag{4}
\]

We will prove that \( g v = P v. \) By taking \( x = x_{2n} \) and \( y = v \) in (1), we get
\[
a(S(f x_{2n}, f x_{2n}, g v), S(Q x_{2n}, Q x_{2n}, P v), S(Q x_{2n}, Q x_{2n}, f x_{2n})
\]
\[
, S(P v, P v, g v), S(g v, g v, Q x_{2n}), S(f x_{2n}, f x_{2n}, P v)) \preceq 0.
\]

Taking \( n \to \infty \) and using (3) and (4), we have
\[
a(S(u, u, g v), S(u, u, u), S(u, u, u), S(u, u, g v)
\]
\[
, S(g v, g v, u), S(u, u, u)) \preceq 0.
\]

This implies that
\[
a(S(u, u, g v), 0, 0, S(u, u, g v), 0) \preceq 0,
\]

which is a contradiction to \( a_3. \) Then, we can say that \( |S(u, u, g v)| = 0. \) Then \( g v = u. \) Consequently,
\[
g v = P v = u. \tag{5}
\]

Then, \( u \) is a point of coincidence of the pair \((g, P)\).
By a similar way, since \( g(X) \subseteq Q(X) \), we can show that

\[
f w = Q w = u. \tag{6}
\]

Then, \( u \) is a point of coincidence of the pair \((f, Q)\).

Hence, \( u \in X \) is a common point of coincidence for the four mappings.

To prove the uniqueness of a point of coincidence. Suppose that \( u^* \neq u \) be another point of coincidence of the four mappings. Then, there exists \( v^*, w^* \) such that \( g v^* = P v^* = u^* \) and \( f w^* = Q w^* = u^* \). Putting \( x = w^* \) and \( y = v \) in (1), one can write

\[
a(S(f w^*, f w^*, g v^*), S(Q w^*, Q w^*, P v^*), S(Q w^*, Q w^*, f w^*))\\n\in S(P v^*, P v^*, g v^*), S(g v^*, g v^*, Q w^*), S(f w^*, f w^*, P v^*)) \preceq 0.
\]

This implies that

\[
a(S(u^*, u^*, u), S(u, u^*, u), S(u^*, u^*, u), S(u, u, u)\\n\in S(u, u, u^*), S(u^*, u^*, u) \preceq 0,
\]

that is,

\[
a(S(u, u, u^*), S(u, u, u^*), S(u, u, u^*), S(u, u, u^*)) \preceq 0,
\]

since \( S(u, u, u^*) = S(u^*, u^*, u) \), which is a contradiction to \( \alpha_3 \). Consequently, the pairs \((f, Q)\) and \((g, P)\) have a unique common point of coincidence.

By using (5), (6) and weak compatibility of the pairs \((f, Q)\) and \((g, P)\), we get

\[
f Q w = Q f w \quad \text{and} \quad g P v = P g v. \tag{7}
\]

Then,

\[
f u = Q u \quad \text{and} \quad g u = P u, \tag{8}
\]

with meaning \( u \) is a point of coincidence of the pairs \((f, Q)\) and \((g, P)\).

Now, we prove that \( u \) is a common fixed point of \( f, g, P \) and \( Q \). Putting \( x = u \) and \( y = v \) in (1), we have

\[
a(S(f u, f u, g v), S(Q u, Q u, P v), S(Q u, Q u, f u)\\n\in S(P v, P v, g v), S(g v, g v, Q u), S(f u, f u, P v)) \preceq 0.
\]

Then,

\[
a(S(f u, f u, u), S(Q u, Q u, u), S(Q u, Q u, f u)\\n\in S(u, u, u), S(u, u, Qu), S(f u, f u, u)) \preceq 0.
\]

This implies that
Theorem 3.2. The conclusion of Theorem 3.1 remains true if completeness of $g(X)$ is replaced by the completeness of one of the subspaces $P(X), Q(X), f(X)$ or $g(X)$.

The following theorem is a new version of Theorem 3.1 under generalized contractive condition.

Theorem 3.3. Let $f, g, P$ and $Q$ be four self-mappings on a complete complex valued $S$–metric space $(X, S)$ such that $f(X) \subseteq P(X)$ and $g(X) \subseteq Q(X)$. Assume that there exists $a \in F$ such that for all $x, y \in X, x \neq y$,

$$a(S(fx, fy, gy), S(Qx, Qx, Py), S(Qx, Qx, fx), S(Py, Py, gy))$$

$$S^2(Qx, Qy, gy) + S^2(Py, Py, fx) \geq 0. \quad (9)$$

If $f(X) \cup g(X)$ is complete subspace of $X$, then the pairs $(f, Q)$ and $(g, P)$ have a unique common point of coincidence. Moreover, if the pairs $(f, Q)$ and $(g, P)$ are weakly compatible, then the four mappings have a unique common fixed point.

Proof. Let $x_0$ be arbitrary points in $X$. Since $f(X) \subseteq P(X)$ and $g(X) \subseteq Q(X)$, then we can define two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ as (2).

Since $\{y_n\} \subseteq f(X) \cup g(X)$. Now, we show that $\{y_n\}$ is a Cauchy sequence. Taking $x = x_{2n}$ and $y = x_{2n+1}$, we get
Now, by a similar way (step by step) of the proof of Theorem 3.1, we can finish the proof.

One of the following:

Corollary 3.1. The end of Theorem 3.1 and 3.3 remains true, if we replace an implicit relation (1) by any where

\[ R. A. Rashwan, H. A. Hammad and M. G. Mahmoud \]

This implies that

\[ \alpha(\mathbf{S}(\mathbf{x}_{2n}, \mathbf{f}_{2n}, \mathbf{Q}_{2n}), \mathbf{S}(\mathbf{Q}_{2n}, \mathbf{Q}_{2n}, \mathbf{P}_{2n+1}) \]

that is,

\[ a(\mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+2}), \mathbf{S}(\mathbf{y}_{2n}, \mathbf{y}_{2n}, \mathbf{y}_{2n+1}), \mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}) \]

\[ \mathbf{S}^{2}(\mathbf{Q}_{2n}, \mathbf{Q}_{2n}, \mathbf{Q}_{2n+1}) + \mathbf{S}^{2}(\mathbf{P}_{2n+1}, \mathbf{P}_{2n+1}, \mathbf{f}_{2n}) \]

\[ \mathbf{S}^{2}(\mathbf{Q}_{2n}, \mathbf{Q}_{2n}, \mathbf{Q}_{2n+1}) + \mathbf{S}(\mathbf{P}_{2n+1}, \mathbf{P}_{2n+1}, \mathbf{f}_{2n}) \]

\[ \mathbf{S}(\mathbf{f}_{2n}, \mathbf{f}_{2n}, \mathbf{P}_{2n+1}) \rightleftharpoons 0, \]

This implies that

\[ a(\mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+2}), \mathbf{S}(\mathbf{y}_{2n}, \mathbf{y}_{2n}, \mathbf{y}_{2n+1}), \mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}) \]

\[ \mathbf{S}^{2}(\mathbf{y}_{2n}, \mathbf{y}_{2n}, \mathbf{y}_{2n+2}) + \mathbf{S}^{2}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}) \]

\[ \mathbf{S}(\mathbf{y}_{2n}, \mathbf{y}_{2n}, \mathbf{y}_{2n+2}) + \mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}) \]

\[ \mathbf{S}(\mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}, \mathbf{y}_{2n+1}) \rightleftharpoons 0. \]

Now, by a similar way (step by step) of the proof of Theorem 3.1, we can finish the proof.

In view of Examples 2.2-2.22 in paper [1], we get the following results which yielding new contraction conditions in $S$—complex valued metric spaces.

**Corollary 3.1.** The end of Theorem 3.1 and 3.3 remains true, if we replace an implicit relation (1) by any one of the following:

(i) $\mathbf{S}(\mathbf{f}, \mathbf{f}, \mathbf{g}, \mathbf{y}) \rightleftharpoons \rho_1(\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})) + \rho_2(\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})) \frac{\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{f}, \mathbf{f}) \mathbf{S}(\mathbf{P}, \mathbf{P}, \mathbf{g}, \mathbf{y})}{1 + \mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})}$

where $\rho_1, \rho_2 : \mathbb{C}_+ \rightarrow [0, 1)$ are given upper semi-continuous mappings such that $\rho_1(z) + \rho_2(z) < 1$ for all $z \in \mathbb{C}_+$.

(ii) $\mathbf{S}(\mathbf{f}, \mathbf{f}, \mathbf{g}, \mathbf{y}) \rightleftharpoons \rho_1(\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})) \mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P}) + \rho_2(\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})) \frac{\mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P}, \mathbf{P}, \mathbf{g}, \mathbf{y})}{1 + \mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})}$

where $\rho_1, \rho_2 : \mathbb{C}_+ \rightarrow [0, 1)$ are given upper semi-continuous mappings such that $\rho_1(z) + \rho_2(z) < 1$ for all $z \in \mathbb{C}_+$.

(iii) $\mathbf{S}(\mathbf{f}, \mathbf{f}, \mathbf{g}, \mathbf{y}) \rightleftharpoons \psi \left( \frac{\mathbf{S}(\mathbf{P}, \mathbf{P}, \mathbf{g}, \mathbf{y}) \mathbf{S}(\mathbf{g}, \mathbf{g}, \mathbf{g}, \mathbf{Q}) \mathbf{S}(\mathbf{f}, \mathbf{f}, \mathbf{P})}{1 + \mathbf{S}(\mathbf{Q}, \mathbf{Q}, \mathbf{P})} \right)$,

where $\psi \in \Psi$ with $\psi(z_1) \rightleftharpoons \psi(z_2) \iff z_1 \leq z_2$. 
(iv) $S(fx, fx, gy) \preceq \delta S(Qx, Qx, Py) + \sigma S(Qx, Qx, fx). S(Py, Py, gy) \over 1 + S(Qx, Qx, fx) + S(Py, Py, gy)$.

where $\delta, \sigma \in \mathbb{R}^+$ such that $\delta + \sigma < 1$.

(v) $S(fx, fx, gy) \preceq \begin{cases} 
\lambda S(Qx, Qx, fx) + \mu S(Qx, Qx, fx) & \text{if } \Delta \neq 0, \\
0 & \text{if } \Delta = 0
\end{cases}$

where $\Delta = S(Qx, Qx, fx) + S(Py, Py, gy)$ and $\lambda, \mu \in \mathbb{R}^+$ such that $\lambda + \mu < 1$.

(vii) $S(fx, fx, gy) \preceq \begin{cases} 
\mu_1 S(Qx, Qx, Py) + \mu_2 S(Qx, Qx, fx) & \text{if } \Delta \neq 0, \\
\mu_3 S(Qx, Qx, Py) + \mu_4 S(Qx, Qx, fx) & \text{if } \Delta = 0
\end{cases}$

where $\Delta = S(Qx, Qx, Py) + S(Qx, Qx, fx) + S(Py, Py, gy)$ and $\mu_i \in \mathbb{R}^+$, $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{4} \mu_i < 1$.

(viii) $S(fx, fx, gy) \preceq \alpha S(Qx, Qx, fx) + \beta \left( S(Qx, Qx, Py)(1 + S(Qx, Qx, fx)) \over 1 + S(Qx, Qx, fx) + S(Py, Py, gy) \right)$.

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$.

(ix) $S(fx, fx, gy) \preceq \alpha_1 S(Qx, Qx, Py) + \alpha_2 (S(Qx, Qx, fx) + S(Py, Py, gy)) + \alpha_3 (S(gy, gy, Qx) + S(fx, fx, Py)) + \alpha_4 \left( S(Py, Py, gy)(1 + S(Qx, Qx, fx)) \over 1 + S(Qx, Qx, Py) + S(Qx, Qx, fx) \right) + \alpha_5 \left( S(Qx, Qx, fx)(1 + S(Py, Py, gy)) \over 1 + S(Qx, Qx, Py) + S(Py, Py, gy) \right) + \alpha_6 S(gy, gy, Qx)$.
where $a_i \in \mathbb{R}^+$, $i = 1, 2, \ldots, 6$, such that $a_1 + 2a_2 + a_3 + a_4 + a_5 + 3a_6 < 1$.

\[(x)\]
\[
S(fx, fx, gy) \preceq \begin{cases} 
\frac{\lambda \Delta}{\alpha} & \text{if } \Delta \neq 0, \\
0 & \text{if } \Delta = 0
\end{cases},
\]
where $\Delta = S(Qx, Qx, Py) + S(Qx, Qx, fx) + S(Py, Py, gy)$,
\[
\nabla = S(Qx, Qx, fx).S(Py, Py, gy) + S(Qx, Qx, Py).S(Py, Py, gy) + S(Qx, Qx, fx).S(Py, Py, gy)
\]
and $\lambda \in \mathbb{R}^+$ such that $\lambda < \frac{1}{4}$.

\[(xi)\]
\[
S(fx, fx, gy) \preceq a_1 S(Qx, Qx, fx) + a_2 \max\{S(Qx, Qx, Py) \, , \, S(Qx, Qx, fx) \, , \, S(fx, fx, Py)\}
\]
\[
+ a_3 \max\{S(Qx, Qx, fx), S(gy, gy, Qx)\},
\]
where $a_i \in \mathbb{R}^+$, $i = 1, 2, 3$ such that $a_1 + a_2 + 3a_3 < 1$.

\[(xii)\]
\[
S(fx, fx, gy) \preceq a \max\{S(Qx, Qx, Py), S(Qx, Qx, fx) 
\]
\[
\, , S(Py, Py, gy), S(gy, gy, Qx), S(fx, fx, Py)\},
\]
where $a \in \mathbb{R}^+$ such that $a < \frac{1}{4}$.

\[(xiii)\]
\[
a_1 S(fx, fx, gy) \preceq a_2 S(Qx, Qx, Py) + a_3 S(Qx, Qx, fx)
\]
\[
+ a_4 S(Py, Py, gy) + a_5 S(gy, gy, Qx),
\]
where $a_i \in \mathbb{R}^+$, $i = 1, 2, \ldots, 5$ such that $a_2 + a_3 + a_4 + 3a_5 < a_1$ and $a_1 > 0$. 
(xiv) \[ S(fx, fx, gy) \preceq \alpha \max \{S(Qx, Qx, fx) + S(Py, Py, gy), \] 
\[ \quad S(gy, gy, Qx) + S(fx, fx, Py) \}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{1}{3} \).

(xv) \[ S(fx, fx, gy) \preceq \alpha \max \{S(Qx, Qx, Py), S(Qx, Qx, fx) \] 
\[ \quad + S(Py, Py, gy), S(gy, gy, Qx) + S(fx, fx, Py) \}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{1}{3} \).

(xvi) \[ S(fx, fx, gy) \preceq \alpha \max \{S(Qx, Qx, Py), S(Qx, Qx, fx) \] 
\[ \quad + S(Py, Py, gy), S(gy, gy, Qx) + S(fx, fx, Py) \}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{1}{3} \).

(xvii) \[ S(fx, fx, gy) \preceq \alpha \max \{S(Qx, Qx, Py), S(gy, gy, Qx) + S(fx, fx, Py) \] 
\[ \quad + \alpha \max \{S(Qx, Qx, Py), S(gy, gy, Qx) \}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{1}{4} \).

(xviii) \[ S(fx, fx, gy) \preceq \alpha \max \left\{ \frac{2S(Qx, Qx, Py) + S(gy, gy, Qx)}{2S(Qx, Qx, Py) + S(Py, Py, gy)}, \right. \]
\[ \left. \quad \frac{2S(Qx, Qx, Py) + S(Py, Py, gy)}{2S(Qx, Qx, Py) + S(fx, fx, Py)} \right\}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{2}{5} \).

(xix) \[ S(fx, fx, gy) \preceq \alpha \max \left\{ \frac{S(Qx, Qx, Py) + S(gy, gy, Qx)}{2}, \right. \]
\[ \left. \quad \frac{S(Py, Py, gy) + S(fx, fx, Py)}{2} \right\}, \]
where \( \alpha \in \mathbb{R}^+ \) such that \( \alpha < \frac{1}{2} \).
(xx)

\[ S(fx, fx, gy) \preceq a \max \{ S(Qx, Qx, Py), S(Qx, Qx, fx), S(Py, Py, gy), S(gy, gy, Qx), S(fx, fx, Py) \} \]
\[ + \beta (S(gy, gy, Qx) + S(fx, fx, Py)), \]

where \( a, \beta \in \mathbb{R}^+ \) such that \( a + \beta < \frac{1}{3} \).

\[(xxi)\]

\[ S(fx, fx, gy) \preceq a \max \{ S(Qx, Qx, Py) \]
\[ \cdot \frac{S(Qx, Qx, fx) + S(Py, Py, gy)}{2}, \frac{S(gy, gy, Qx) + S(fx, fx, Py)}{2} \}, \]

where \( a \in [0, 1) \).

4 Applications

In this section, we use the contraction condition (i) of the corollary 3.1 to prove the existence and uniqueness of common solution for the system of Hammerstein integral equations:

\[ x(t) = \psi_j(t) + \int_a^b k_j(t, s) f_j(s, x(s)) \, ds, \tag{10} \]

where \( t \in [a, b] \subseteq \mathbb{R}, x, \psi_j \in X = C([a, b], \mathbb{R}^n), k_j : [a, b] \times [a, b] \longrightarrow \mathbb{R}^n \) and \( f_j : [a, b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, j = 1, 2. \)

Also, we apply the contraction condition (iii) of Corollary 3.1 to obtain the existence and uniqueness of common solution for the system of Urysohn integral equations:

\[ x(t) = \psi_j(t) + \int_a^b k_j(t, s) \, ds, \tag{11} \]

where \( t \in [a, b] \subseteq \mathbb{R}, x, \psi_j \in X, k_j : [a, b] \times [a, b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \) and \( j = 1, 2. \)

We use in this section, the following symbols:

\[ H_j(x(t)) = \int_a^b k_j(t, s) f_j(s, x(s)) \, ds, \quad j = 1, 2; \]
\[ R_j(x(t)) = \int_a^b k_j(t, s, x(s)) \, ds, \quad j = 1, 2; \]
\[ M_{xy}(t) = 2 \| H_j(x(t)) + \phi_1(t) - H_j(y(t)) - \psi_2(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ N_{xy}(t) = 2 \| x(t) - y(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ M_{xy}(t) = 2 \| R_j(x(t)) + \phi_1(t) - R_j(y(t)) - \psi_2(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ A_{xy}(t) = 2 \| R_2(y(t)) + \phi_2(t) - y(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ B_{xy}(t) = 2 \| R_2(y(t)) + \phi_2(t) - y(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ C_{xy}(t) = 2 \| R_1(x(t)) + \phi_1(t) - y(t) \|_n \sqrt{1 + \beta^3} \| e^{\cot^{-1} \beta}; \]
\[ D_{xy}(t) = 2 \left( \frac{\|H_1(x(t)) + \psi_1(t) - x(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}}}{1 + \sup_{t \in [a,b]} N_{xy}(t)} \right) \]
\[ \times 2 \|H_2(y(t)) + \psi_2(t) - y(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}} \]
\[ E_{xy}(t) = \left( \frac{A_{xy}(t)b_{xy}(t)c_{xy}(t)}{1 + \sup_{t \in [a,b]} N_{xy}(t)} \right) \]

Now, our main theorems become valid for viewing.

**Theorem 4.1.** Let \( X = C([a,b], \mathbb{R}^n); a > 0 \). Consider the system (10) and \( k_1, k_2, f_1 \) and \( f_2 \) are such that \( H_1(x), H_2(x) \in X \) for all \( x \in X \). If there exists \( \rho_1, \rho_2 : C_+ \rightarrow [0,1) \) such that for each \( t \in C, x, y \in X \) and \( t \in [0,1] \), we get

\[ (c_1) \quad \rho_1(z) + \rho_2(z) < 1, \]

\[ (c_2) \quad M_{xy}(t) \geq \rho_1 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) N_{xy}(t) \]
\[ + \rho_2 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) D_{xy}(t). \]

Then, the system of integral equations (10) have a unique common solution.

**Proof.** Define S-metric \( S : X \times X \rightarrow \mathbb{R} \) by

\[ S(x, y) = S(x, x, y) = 2 \sup_{t \in [a,b]} \|x(t) - y(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}}, \quad (12) \]

then, \( (X, S) \) is a complete complex S-metric space.

Define two mappings \( f, g : X \rightarrow X \) as follows:

\[ f(x(t)) = \psi_1(t) + H_1(x(t)) = \psi_1(t) + \int_a^b k_1(t, s) f_1(s, x(s)) \, ds, \]
\[ g(x(t)) = \psi_2(t) + H_2(x(t)) = \psi_2(t) + \int_a^b k_2(t, s) f_2(s, x(s)) \, ds. \]

Let \( x, y \in X \). By the simple calculation, we have

\[ S(fx, fx, xy) = 2 \sup_{t \in [a,b]} \|H_1(x(t)) + \psi_1(t) - H_2(y(t)) - \psi_2(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}}, \]
\[ S(x, x, fx) = 2 \sup_{t \in [a,b]} \|H_1(x(t)) + \psi_1(t) - x(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}}, \]
\[ S(y, y, xy) = 2 \sup_{t \in [a,b]} \|H_2(y(t)) + \psi_2(t) - y(t)\|_\infty \sqrt{T + p^* e^{\cot^{-1}b}}, \]

Applying assumption \((c_2)\), for each \( t \in [a,b] \), we can write

\[ M_{xy}(t) \geq \rho_1 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) N_{xy}(t) + \rho_2 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) D_{xy}(t), \]

\[ \geq \rho_1 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) \sup_{t \in [a,b]} N_{xy}(t) \]
\[ + \rho_2 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) \sup_{t \in [a,b]} D_{xy}(t), \]

which leads to

\[ \sup_{t \in [a,b]} M_{xy}(t) \geq \rho_1 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) \sup_{t \in [a,b]} N_{xy}(t) \]
\[ + \rho_2 \left( \sup_{t \in [a,b]} N_{xy}(t) \right) \sup_{t \in [a,b]} D_{xy}(t), \]
which yielding
\[ S(fx, fx, gy) \preceq \rho_1 (S(x, x, y)) S(x, x, y) + \rho_2 (S(x, x, y)) \frac{S(x, x, fx) S(y, y, gy)}{1 + S(x, x, y)}. \]

Thus, the contraction condition (i) of Corollary 3.1 is satisfied with \( P = Q = I \) (where \( I \) is the identity mapping). So, according to Theorem 3.1, the system (10) has a unique solution.

**Theorem 4.2.** Let \( X = C([a, b], \mathbb{R}^n); a > 0 \). Consider the system of integral equation (11) and \( k_1, k_2 \) are such that \( R_1(x), R_2(x) \in X \). If for each \( x, y \in X \) and \( t \in [a, b] \), we have
\[
M_{xy}^*(t) \preceq E_{xy}(t). \tag{13}
\]

Then, the system of integral equations (11) have a unique common solution.

**Proof.** Define the some S-metric (12) and define two mappings \( f, g : X \to X \) as the following:
\[
f(x(t)) = \varphi_1(t) + R_1(x(t)) = \varphi_1(t) + \int_a^b k_1(t, s, x(s)) \, ds;
g(x(t)) = \varphi_2(t) + R_2(x(t)) = \varphi_2(t) + \int_a^b k_2(t, s, x(s)) \, ds.
\]

For \( x, y \in X \), by the same manner of Theorem 4.1, we can calculate
\[
S(fx, fx, gy) = 2 \sup_{t \in [a, b]} \| R_1(x(t)) - \varphi_1(t) \|_\infty \sqrt{T + p} e^{n \cdot t};
\]
\[
S(gy, gy, x) = 2 \sup_{t \in [a, b]} \| R_2(y(t)) + \varphi_2(t) - x(t) \|_\infty \sqrt{T + p} e^{n \cdot t};
\]
\[
S(fx, fx, y) = 2 \sup_{t \in [a, b]} \| R_1(x(t)) + \varphi_1(t) - y(t) \|_\infty \sqrt{T + p} e^{n \cdot t};.
\]

From the assumption (13) for each \( t \in [a, b] \)
\[
M_{xy}^*(t) \preceq E_{xy}(t) \preceq \sup_{t \in [a, b]} E_{xy}(t),
\]

which yields
\[
\sup_{t \in [a, b]} M_{xy}^*(t) \preceq \sup_{t \in [a, b]} E_{xy}(t),
\]
or equivalently,
\[
\psi\left(\sup_{t \in [a, b]} M_{xy}^*(t)\right) \preceq \psi\left(\sup_{t \in [a, b]} E_{xy}(t)\right),
\]
hence,
\[
\psi\left(S(fx, fx, gy)\right) \preceq \psi\left(S(gy, gy, x) S(fx, fx, y) / (1 + S(x, x, y))\right).
\]

Thus, all conditions of Theorem 3.1 corresponding to contraction (iii) of corollary 3.1 with \( P = Q = I \) are hold. Then, the nonlinear integral equations (11) have a unique common solution.

**Acknowledgements**

The author wishes to thank the chief editor and the anonymous referees for their valuable comments and suggestions, which helped us very much in presenting and improving the original version of the paper.
References


