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On the qualitative properties of functional integral equations with abstract Volterra operators

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Abstract: Using the weakly Picard operators technique we establish existence, data dependence and comparison results of solutions for a functional integral equation with abstract Volterra operators. Some examples which show the importance of our results are also included.

Keywords: Functional integral equation, weakly Picard operators, data dependence and abstract Volterra operator.

MSC: 47H10, 34K05.

1 Introduction and preliminaries

It is well known that functional integral equations of different types find numerous applications in describing real-world problems which appear in mechanics, physics, engineering, biology, see for example [2], [3]-[5] and [7].

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The purpose of this paper is to study existence, data dependence and comparison results for the solutions of the functional integral equation of the form

$$x(t) = F(t, h(x)(t), x(0)) + \int_0^t K(x)(s)ds, \ t \in [0, b]. \tag{1.1}$$

using the weakly Picard operators technique. The theory of Picard operators was introduced by I. A. Rus (see [15]-[16] and their references) to study problems related to fixed point theory. This abstract approach is used by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

Our results extend and improve corresponding theorems in the existing literature (see, e.g. [17], [18], [9], [11], [12] and [6]). Some properties of the solutions to differential and integral equations with abstract Volterra operators were studied, for example, in [1], [8] and [10].

In this paper we use the terminologies and notations from [15]-[16]. Let us recall now some essential definitions and fundamental results.

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by $A^0 = 1_X$, $A^1 = A$, $A^{n+1} := A \circ A^n, n \in \mathbb{N}$ the iterates of the operator A;

We also use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subsets of A.

We begin with the definitions of a Picard and weakly Picard operator.

Definition 1.1. *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 1.2. *Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 1.3. *If A is a weakly Picard operator then we consider the operator A^∞ defined by $A^\infty : X \to X$, $A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$.*

Remark 1.4. *It is clear that $A^\infty(X) = F_A$.*

In the sequel, the following results are useful for some of the proofs in the paper.

Lemma 1.5. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that A is WPO and A is increasing. Then, the operator A^∞ is increasing.*

Lemma 1.6. *(Abstract Gronwall lemma) Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that A is WPO and A is increasing. Then:*

(a) $x \leq A(x) \Longrightarrow x \leq A^\infty(x)$;

(b) $x \geq A(x) \Longrightarrow x \geq A^\infty(x)$.

Lemma 1.7. *(Abstract comparison lemma) Let $(X, d, \leq)$ an ordered metric space and $A, B, C : X \to X$ be such that: (i) the operators A, B, C are WPOs; (ii) $A \leq B \leq C$; (iii) the operator B is increasing. Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.*

Another important notion is

Definition 1.8. *Let (X, d) be a metric space, $A : X \to X$ be a weakly Picard operator and $c \in \mathbb{R}_+^*$. The operator A is c-weakly Picard operator iff*

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall x \in X.$$

For the c-POs and c-WPOs we have

Lemma 1.9. *Let (X, d) be a metric space, $A, B : X \to X$ be two operators. We suppose that:*

(i) the operators A and B are c-WPOs;

(ii) there exists $\eta \in \mathbb{R}_+^*$ such that $d(A(x), B(x)) \leq \eta$, $\forall x \in X$.

Then $H_d(F_A, F_B) \leq c\eta$, where H_d stands for the Pompeiu-Hausdorff functional with respect to d.

We note that most operators in the space of nonexpansive operators are (weakly) Picard operators. See, for example, the two papers by S. Reich and A. J. Zaslavski [13] and [14]. For some examples of WPOs see [15]-[18].

2 Main results

Let $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ be a Banach space. We consider the equation (1.1) in the following conditions:

(C_1) $F \in C([0, b] \times \mathbb{B} \times \mathbb{B}, \mathbb{B})$;

(C_2) $h : C([0, b], \mathbb{B}) \to \mathbb{B}$ is an abstract Volterra operator and there exists $L_h > 0$ such that

$$|h(x)(t) - h(y)(t)| \leq L_h |x(t) - y(t)|,$$

$\forall x, y \in C([0, b], \mathbb{B}),\ t \in [0, b]$;

(C_3) $K : C([0, b], \mathbb{B}) \to \mathbb{B}$ is an abstract Volterra operator and there exists $L_K > 0$ such that

$$|K(x)(t) - K(y)(t)| \leq L_K |x(t) - y(t)|,$$

$\forall x, y \in C([0, b], \mathbb{B}),\ t \in [0, b]$;

(C_4) there exists $0 < L_F < 1$ such that

$$|F(t, u, \lambda) - F(t, v, \lambda)| \leq L_F |u - v|,$$

$\forall u, v \in \mathbb{B},\ t, \lambda \in [0, b]$;

(C_5) $F(0, h(x)(0), x(0)) = x(0)$;

(C_6) there exists $\beta \in \mathbb{B}$ such that

$$h(x)(0) = \beta, \ \forall x \in C([0,b], \mathbb{B}).$$

With respect to the equation (1.1) we consider the equation (in $\lambda \in \mathbb{B}$)

$$\lambda = F(0, \beta, \lambda). \tag{2.1}$$

Let S_F be the solution set of the equation (2.1).

In what follows we consider the space $X := (C([0,b], \mathbb{B}), \|\cdot\|_\tau)$, where $\|\cdot\|_\tau$ is the Bielecki norm defined by $\|x\|_\tau = \max |x(t)| e^{-\tau t}$, $\tau > 0$, and the operator $A : X \to X$ be defined by

$$A(x)(t) = F(t, h(x)(t), x(0)) + \int_0^t K(x)(s)ds, \ t \in [0,b].$$

Let $X_\lambda = \{x \in C([0,b], \mathbb{B}) | \ x(0) = \lambda\}$. Notice that $X = \underset{\lambda \in \mathbb{B}}{\cup} X_\lambda$ is a partition of X and we have the following lemma (see [17]).

Lemma 2.1. *We have*

(i) If $x \in F_A$, then $x(0) \in S_F$;

(ii) $F_A \cap X_\lambda \neq \emptyset \Rightarrow \lambda \in S_F$.

Our first main result is the following. We aim to prove an existence theorem for the solution of equation (1.1).

Theorem 2.2. *If the conditions $(C_1) - (C_6)$ are satisfied, then equation (1.1) has a solution in $C([0,b], \mathbb{B})$. Moreover, $A|_{\underset{\lambda \in S_F}{\cup} X_\lambda} : \underset{\lambda \in S_F}{\cup} X_\lambda \to \underset{\lambda \in S_F}{\cup} X_\lambda$ is a WPO and card $F_A =$ card S_F.*

Proof. We denote by $A_\lambda := A|_{X_\lambda} : X_\lambda \to X_\lambda$, $\lambda \in S_F$. From $(C_1) - (C_6)$ we have

$$\begin{aligned} |A_\lambda(x)(t) - A_\lambda(y)(t)| &\leq L_F L_h \|x-y\| e^{\tau t} + \tfrac{L_K}{\tau} \|x-y\| e^{\tau t} \\ &\leq (L_F L_h + \tfrac{L_K}{\tau}) \|x-y\| e^{\tau t}, \end{aligned}$$

and therefore

$$\|A_\lambda(x) - A_\lambda(y)\| \leq \left(L_F L_h + \tfrac{L_K}{\tau}\right) \|x-y\|, \ \forall x,y \in C([0,b], \mathbb{B}).$$

For a suitable choice of τ, the operator $A|_{X_\lambda}$ is a contraction with respect to $\|\cdot\|_\tau$. From the fact that A_λ with $\lambda \in S_F$ is PO and from Lemma 2.1 we have that *card* $F_A =$ *card* S_F. Moreover, from the characterization theorem of WPOs (see [15]) we get that A is a WPO. □

Next we shall study some comparison results for the solution to the equation (1.1).

Theorem 2.3. *We consider the equation* (1.1) *such that all the assumptions to the Theorem 2.2 hold. In addition, we suppose that:*

(i) $\mathbb{B}$ is an ordered Banach space;

(ii) the operators $F(t,\cdot,\cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, $h, K : \mathbb{B} \to \mathbb{B}$ are increasing, $\forall t \in [0,b]$;

Let x and y be two solutions to the equation (1.1). *If $x(0) \leq y(0)$ then $x(t) \leq y(t)$ for all $t \in [0,b]$.*

Proof. We remark that $x \in X_{x(0)}$ and $y \in X_{y(0)}$. If $x \in \mathbb{B}$, then we denote by $\widetilde{x}$ the constant function

$$\widetilde{x} : \mathbb{B} \to \mathbb{B},\ \widetilde{x}(t) = x,\ \forall t \in [0,b].$$

From $\widetilde{x}(0) \in X_{x(0)}$ and $\widetilde{y}(0) \in X_{y(0)}$ we have that

$$x = A^{\infty}(\widetilde{x}(0)),\ y = A^{\infty}(\widetilde{y}(0)).$$

From $(C_1) - (C_6)$ the operator A is a WPO and from (ii) the operator A is increasing. Applying the Lemma 1.5 we obtain that A^{∞} is increasing. From the Theorem 2.2 we have that $A(X) \subset X$. $A|_X$ is a contraction and since $\widetilde{x} \in X$ then

$$A^{\infty}(\widetilde{x}) = A^{\infty}(x),\ \forall x \in X.$$

Let $x \leq A(x)$, since A is increasing, from the Gronwall lemma (Lemma 1.6) we get $x \leq A^{\infty}(x)$. Also, $x, \widetilde{x}(0) \in X_{x(0)}$, so $A^{\infty}(x) = A^{\infty}(\widetilde{x}(0))$. But $x(0) \leq y(0)$, A^{∞} is increasing and $A^{\infty}(\widetilde{y}(0)) = A^{\infty}(y) = y$. So,

$$x \leq A^{\infty}(x) = A^{\infty}(\widetilde{x}(0)) \leq A^{\infty}(\widetilde{y}(0)) = y.$$

So the proof is completed. □

In the following part of this section we study the order preserving property of the equation (1.1) with respect to K. For this we use the Lemma 1.7.

Theorem 2.4. *Let $F_i \in C([0,b] \times \mathbb{B}, \mathbb{B})$, $h, K_i \in C([0,b], \mathbb{B}), i \in \{1,2,3\}$ be as in the Theorem 2.2. Furthermore, we suppose that:*

(i) $F_1 \leq F_2 \leq F_3$, $K_1 \leq K_2 \leq K_3$;

(ii) *the operators $F_2(t,\cdot,\cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ and $h, K_2 : \mathbb{B} \to \mathbb{B}$ are increasing;*

(iii) $S_{F_1} = S_{F_2} = S_{F_3}$.

If $x_i \in C([0,b], \mathbb{B})$ is a solution to the equation (1.1) *corresponding to F_i and $K_i, i \in \{1,2,3\}$, then*

$$x_1(0) \leq x_2(0) \leq x_3(0) \textit{ imply that } x_1(t) \leq x_2(t) \leq x_3(t),\ \forall t \in [0,b].$$

Proof. Applying the Theorem 2.2 we have that the operators $A_i, i \in \{1,2,3\}$ are WPOs. From the conditions (i) and (ii) to the theorem, follows that the operator A_2 is monotone increasing and $A_1 \leq A_2 \leq A_3$.

Let now $\widetilde{x}_i(0) \in C([0,b], \mathbb{B})$ be defined by $\widetilde{x}_i(0)(t) = x_i(0)$, $\forall t \in [0,b]$. It is clear that the following inequalities between the defined functions hold:

$$\widetilde{x}_1(0)(t) \leq \widetilde{x}_2(0)(t) \leq \widetilde{x}_3(0)(t),\ \forall t \in [0,b].$$

Now we apply the Lemma 1.7 to the above inequalities and we have that

$$A_1^{\infty}(\widetilde{x}_1(0)) \leq A_2^{\infty}(\widetilde{x}_2(0)) \leq A_3^{\infty}(\widetilde{x}_3(0)).$$

But $x_i = A_i^{\infty}(\widetilde{x}_i(0)), i \in \{1,2,3\}$ and therefore, from the Lemma 1.7, we get that $x_1(t) \leq x_2(t) \leq x_3(t)$, $\forall t \in [0,b]$. □

In the last part of this section we present a data dependence result for the solutions to two similar problems with different parameters. We consider the following functional integral equations

$$x(t) = F_i(t, h(x)(t), x(0)) + \int_0^t K_i(x)(s)ds,\ t \in [0, b], i \in \{1, 2\}.$$

We denote by $A_i : X \to X$,

$$A_i(x)(t) = F_i(t, h(x)(t), x(0)) + \int_0^t K_i(x)(s)ds,\ t \in [0, b], i \in \{1, 2\}.$$

We have

Theorem 2.5. *We consider $F_i, K_i,\ i \in \{1, 2\}$ satisfying the conditions $(C_1) - (C_6)$. In addition, we suppose*

(i) there exists $\eta_1 > 0$ such that $|F_1(t, u_1, u_2) - F_2(t, u_1, u_2)| \le \eta_1, \forall t \in [0, b], u_1, u_2 \in \mathbb{B}$;

(ii) there exists $\eta_2 > 0$ such that $|K_1(x)(s) - K_2(x)(s)| \le \eta_2, \forall s \in [0, b], x \in C([0, b], \mathbb{B})$;

Then

$$H_{\|\cdot\|_B}(F_{A_1}, F_{A_2}) \le (\eta_1 + b\eta_2) \max\left\{\frac{1}{1-L_1}, \frac{1}{1-L_2}\right\},$$

where $L_i := L_{F_i} L_h + \frac{L_{K_i}}{\tau},\ i \in \{1, 2\}$, for τ suitable selected and $H_{\|\cdot\|_B}$ denotes the Pompeiu-Housdorff functional with respect to $\|\cdot\|_\tau$.

Proof. From the Theorem 2.2 we have that $A_i|_{\underset{\lambda\in S_{F_i}}{\cup} X_\lambda} : \underset{\lambda\in S_{F_i}}{\cup} X_\lambda \to \underset{\lambda\in S_{F_i}}{\cup} X_\lambda, i \in \{1, 2\}$ are WPOs. Moreover, $A_i|_{X_\lambda}$ is a contraction, with constant $L_i = L_{F_i} L_h + \frac{L_{K_i}}{\tau},\ i \in \{1, 2\}$, with respect to $\|\cdot\|_\tau$ for a suitable choice of τ. Therefore $A_i|_{\underset{\lambda\in S_{F_i}}{\cup} X_\lambda}$ is $c_i - WPO$, with $c_i = \frac{1}{1-L_i}$. On the other hand we have that

$$\begin{aligned}
|A_1(x)(t) - A_2(x)(t)| &\le |F_1(t, h(x)(t), x(0)) - F_2(t, h(x)(t), x(0))| + \\
&+ \int_0^t |K_1(x)(s) - K_2(x)(s)|\, ds \\
&\le \eta_1 + b\eta_2,\ \forall x \in C([0, b], \mathbb{B}), t \in [0, b].
\end{aligned}$$

The conclusion follows from the Lemma 1.9. □

3 Special cases

In this section, we give some examples of some functional-integral equations considered in the applied problems of nonlinear analysis which are particular cases of equation (1.1).

Example 3.1.

$$x(t) = x(0) + \int_0^t K(x)(s)ds,\ t \in [0, b]. \tag{3.1}$$

In this case, the conditions $(C_1) - (C_6)$ become:

(C_3) $K : C([0, b], \mathbb{B}) \to \mathbb{B}$ is an abstract Volterra operator and there exists $L_K > 0$ such that

$$|K(x)(t) - K(y)(t)| \le L_K |x(t) - y(t)|,$$

$\forall x, y \in C([0, b], \mathbb{B}),\ t \in [0, b]$;

Let S_1 be the solution set to the equation (3.1). Notice that in this condition we have that $S_1 = \mathbb{B}$ and the integral equation has an infinite number of solutions. Also one can apply the theorems 2.2, 2.3, 2.4 and 2.5 for the study of existence and uniqueness, comparison results, order preserving property and data dependence of the solution to the equation (3.1).

Example 3.2.

$$x(t) = h(x)(t) + \int_0^t K(x)(s)ds, \ t \in [0,b]. \tag{3.2}$$

In this case, the conditions $(C_1) - (C_6)$ become:

(C$_2$) $h : C([0,b],\mathbb{B}) \to \mathbb{B}$ is an abstract Volterra operator and there exists $L_h > 0$ such that

$$|h(x)(t) - h(y)(t)| \leq L_h |x(t) - y(t)|,$$

$\forall x, y \in C([0,b],\mathbb{B}), \ t \in [0,b]$;

(C$_3$) $K : C([0,b],\mathbb{B}) \to \mathbb{B}$ is an abstract Volterra operator and there exists $L_K > 0$ such that

$$|K(x)(t) - K(y)(t)| \leq L_K |x(t) - y(t)|,$$

$\forall x, y \in C([0,b],\mathbb{B}), \ t \in [0,b]$;

(C$_5$) $h(x)(0) = x(0)$.

Let S_2 be the solution set to the equation (3.2). In this case $S_2 = \mathbb{B}$ and, therefore, the integral equation has an infinite number of solutions. Also one can apply the theorems 2.2, 2.3, 2.4 and 2.5 for the study of existence and uniqueness, comparison results, order preserving property and data dependence of the solution to the equation (3.2).

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