Res. Fixed Point Theory Appl.
Volume 2018, Article ID 2018013, 08 pages
eISSN 2581-6047

Results in Fixed Point Theory and Applications

RESEARCH ARTICLE

A Fixed point Theorem Satisfying Integral Type Contraction in Fuzzy Metric Space

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Academic Editor: Supuk Phiangsungnoen

Abstract: In this paper we analyze the existence of fixed points for mappings defined on a complete fuzzy metric space satisfying a contractive condition of integral type. Our main result generalize the fuzzy Banach contraction theorem of V. Gregori and A. Sapena [V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems, 125(2) (2002) 245 − 252]

Keywords: Fuzzy metric space Integral type contraction Fuzzy Banach contraction.

MSC: 54H25, 47H10, 54E50.

1 Introduction

In 1965 L. Zadeh [18] introduced the theory of fuzzy sets. Many authors have introduced the concept of fuzzy metric spaces in different ways. In 1975 Kramosil and Michalek [10] introduced the notion of fuzzy metric space by using continuous t norm. Later on in 1994 George and Veeramani [3] modify the concept of fuzzy metric space introduced by Kramosil and Michalek. They define a Hausdorff topology and have show that every metric induced a fuzzy metric. Form the definition of Cauchy sequence given

DOI: 10.30697/rfpta-2018-013

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Received May 27, 2018; revised October 16, 2018; accepted November 08, 2018.

by Kramosil and Michalek even $\mathbb{R}$ is not complete so George and Veeramani also modify the definition of Cauchy sequence. V. Gregori and A. Sapenal [4] extend the Banach fixed point theorem to fuzzy contractive mapping of complete metric space. In the recent years, many authors had dedicated themselves to the study of fixed point theory in fuzzy metric space (see [5, 6, 7, 11, 12, 13, 14, 15] and reference therein) Branciari [2] gave an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. Afterwards many researchers extended the result of Branciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces, for more details see [1, 9, 8, 16, 17].

2 Preliminaries

Let us firstly recall some basic definitions and known results needed in the sequel.

Definition 2.1. *[10] A binary operation* $* : [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$

Some popular t norm are listed below

(i) $x * y = x \wedge y$

(ii) $x * y = \max(0, x + y - 1)$ (Lukasiewicz t-norm)

$$(iii)\ x * y = \begin{cases} x & \text{if} \quad y = 1 \\ y & \text{if} \quad x = 1; \\ 0 & \text{otherwise} \end{cases}$$

(iv) $x * y = xy$.

A continuous t-norm $*$ is of Hadzic-type if there exist strictly increasing sequence $\{b_n\} \subset (0,1)$ such that $b_n * b_n = b_n$ for all $n \in \mathbb{N}$.

Definition 2.2. *[10] The 3-tuple* $(X, M, *)$ *is said to be fuzzy metric space if X is an arbitrary set,* $*$ *is continuous t-norm, and* $M : X \times X \times [0, \infty) \to [0,1]$ *is a fuzzy set satisfying the following conditions:*

(FM_1) $M(x, y, 0) = 0$,

(FM_2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

(FM_3) $M(x, y, t) = M(y, x, t)$,

(FM_4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM_5) $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is left continious for all $x, y, z \in X$ and $t, s > 0$.Let $X = \mathbb{N}$, and take $a * b = ab$.

Example 2.3. *[4] Define for all $t > 0$,*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space.

Example 2.4. *[4] Let $X = \mathbb{R}$. Define $a * b = ab$ and*

$$M(x, y, t) = \left[\exp\left(\frac{|x - y|}{t}\right)\right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a fuzzy metric space.

George and Veeramani modify the concept of fuzzy metric space as following

Definition 2.5. *[10] The 3-tuple $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, and $M : X \times X \times [0, \infty) \to [0, 1]$ is a fuzzy set satisfying the following conditions: (FM_1) $M(x, y, t) > 0$,*

(FM_2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

(FM_3) $M(x, y, t) = M(y, x, t)$,

(FM_4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM_5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.Let $X = \mathbb{N}$, and take $a * b = ab$.

Example 2.6. *[4] Define for all $t > 0$,*

$$M(x, y, t) = \frac{1}{1 + |x - y|}$$

Then $(X, M, *)$ is a fuzzy metric space. Where $a * b = ab$

Definition 2.7. *[6] A sequence (x_n) in a fuzzy metric space $(X, M, *)$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.*

Definition 2.8. *[6] $\varepsilon \in (0, 1)$ and each $t > 0$ there exist a natural number N such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq N$. A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.*

Definition 2.9. *[6] Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \to X$ is called fuzzy contractive mapping if there exists $k \in (0, 1)$, for all distinct $x, y \in X$ and $t > 0$ such that*

$$\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

Theorem 2.10. *(Fuzzy Banach contraction theorem)[12] . Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \to X$ be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.*

A. Branciari generalized the Banach fixed point theorems as following

Theorem 2.11. *[2] Let (X, d) be a complete metric space, $k \in (0, 1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \varphi(s)\, ds \leq k \int_0^{d(x,y)} \varphi(s)\, ds$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, non negative,and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)\, ds > 0$$

then T has a unique fixed point $z \in X$ such that for each $z \in X$, $\lim_{n\to\infty} T^n x = z$.

3 Main result

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space, $k \in (0, 1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)\, ds \leq k \int_0^{\left(\frac{1}{M(x,y,t)}-1\right)} \varphi(s)\, ds \tag{1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, non negative,and such that for each $\varepsilon > 0$,

$$\int_0^{1-\varepsilon} \varphi(s)\, ds > 0 \tag{2}$$

then T has a unique fixed point $z \in X$ such that for each $z \in X$, $\lim_{n\to\infty} T^n x = z$.

Proof. Let $x \in X$ be an arbitrary point define a sequence $x_n = T^n x$. For each integer $n \geq 1$, using (1)

$$\int_0^{\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)} \varphi(s)\, ds \leq k \int_0^{\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)} \varphi(s)\, ds$$

repeating this process n times we get

$$\int_0^{\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)} \varphi(s)\, ds \leq k^n \int_0^{\left(\frac{1}{M(x_0,x_1,t)}-1\right)} \varphi(s)\, ds$$

Taking limit $n \to \infty$ we obtained

$$\lim_n \int_0^{\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right)} \varphi(s)\, ds = 0$$

which from (2) implise

$$\lim_n \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) = 0$$

which implise

$$\lim_n (M(x_n, x_{n+1}, t)) = 1 \tag{3}$$

Now we have to show that (x_n) is a Cauchy sequence. Suppose that (x_n) is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and sub-sequence (m_p) and (n_p) such that $m_p < n_p < m_{p+1}$ with

$$M(x_{m_p}, x_{n_p}, t) \leq 1 - \varepsilon \qquad M(x_{m_p}, x_{n_p-1}, t) > 1 - \varepsilon \tag{4}$$

by using (3), we get

$$\int_0^{\left(\frac{1}{M(x_{m_p},x_{m_p-1},t)}-1\right)} \varphi(s)\,ds = \int_0^{\left(\frac{1}{M(x_{n_p},x_{m_p-1},t)}-1\right)} \varphi(s)\,ds = 0$$

from triangular inequality and (4),

$$\begin{aligned} M(x_{m_p-1},x_{n_p-1},t) \geq & M(x_{m_p-1},x_{m_p},\frac{t}{2}) * M(x_{m_p},x_{n_p-1},\frac{t}{2}) \\ > & M(x_{m_p-1},x_{m_p},\frac{t}{2}) * (1-\varepsilon) \xrightarrow{p\to\infty} (1-\varepsilon) \end{aligned} \tag{5}$$

Hence

$$\lim_p \int_0^{\left(\frac{1}{M(x_{m_p-1},x_{n_p-1},t)}-1\right)} \varphi(s)\,ds < \int_0^{\frac{\varepsilon}{1-\varepsilon}} \varphi(s)\,ds \tag{6}$$

By using (1),(3) and (6) we have

$$\begin{aligned} \int_0^{\frac{\varepsilon}{1-\varepsilon}} \varphi(s)\,ds \leq & \int_0^{\left(\frac{1}{M(x_{m_p},x_{n_p},t)}-1\right)} \varphi(s)\,ds \\ \leq & k \int_0^{\left(\frac{1}{M(x_{m_p-1},x_{n_p-1},t)}-1\right)} \varphi(s)\,ds \\ < & k \int_0^{\frac{\varepsilon}{1-\varepsilon}} \varphi(s)\,ds \end{aligned}$$

which is a contraction so (x_n) is a Cauchy sequence. Since X is complete so there exist $z \in X$ such that $x_n \to z$. Now, using (1) and taking $n \to \infty$ we get,

$$\begin{aligned} \int_0^{\left(\frac{1}{M(Tz,x_{n+1},t)}-1\right)} \varphi(s)\,ds \leq & k \int_0^{\left(\frac{1}{M(z,x_n,t)}-1\right)} \varphi(s)\,ds \\ \leq & k \int_0^{\left(\frac{1}{M(z,z,t)}-1\right)} \varphi(s)\,ds = 0 \end{aligned}$$

which implies that $M(Tz,z,t) = 1$ and hence $Tz = z$, that is, z is fixed point of T. For uniqueness let us suppose that y and z are two distinct fixed points of T then from (1)

$$\begin{aligned} \int_0^{\left(\frac{1}{M(y,z,t)}-1\right)} \varphi(s)\,ds = & \int_0^{\left(\frac{1}{M(Ty,Tz,t)}-1\right)} \varphi(s)\,ds \\ \leq & k \int_0^{\left(\frac{1}{M(y,z,t)}-1\right)} \varphi(s)\,ds \end{aligned}$$

since $k < 1$ so $\left(\frac{1}{M(y,z,t)} - 1\right) = 0$. Which implise $M(y,z,t) = 1$ or $y = z$. □

Remark 3.2. *The Theorem* 2.10 *is a special case of our main result. That is by putting* $\varphi(s) = 1$ *in the inequality* (1) *for each* $t \geq 0$, *we get the inequality* (2.9). *The example 3.7 shows the generality of our main result.*

Remark 3.3. *We have used the idea of integral fuzzy contraction to generalize Fuzzy Banach contraction theorem, but in a similar way we can generalize other results also related to contractive conditions of some kind, such as the ones contained in [2, 3, 4, 10, 12, 13].*

Remark 3.4. *Theorem 3.1 is not true if we take zero value almost everywhere near zero for the mapping* φ*; following example shows our claim. In a similar way, Theorem 3.1 is not true for the negative value of* φ*, as in Example 3.5.*

Example 3.5. *Let $X = \mathbb{N}$ with fuzzy metric defined by $M(x,y,t) = \frac{1}{1+|x-y|}$. Let $T : X \to X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by*

$$T(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases} \qquad \varphi(s) = \begin{cases} e^{\frac{1}{1-s}} & s \geq 1 \\ 0 & s \in [0,1] \end{cases},$$

Now, since for every $x, y \in X$ and $t \geq 0$, $\left(\frac{1}{M(Tx,Ty,t)} - 1\right) \leq 1$, hence for arbitray $k \in (0,1)$

$$\int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)\, ds \leq \int_0^1 \varphi(s)\, ds = 0 \leq k \int_0^{\left(\frac{1}{M(x,y,t)}-1\right)} \varphi(s)\, ds;$$

Thus (1) satisfied for all $k \in (0,1)$, but T has no fixed point.

Example 3.6. *Let $X = \mathbb{R}^+$ with fuzzy metric defined by $M(x,y,t) = \frac{1}{1+|x-y|}$. Let $T : X \to X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $T(x) = x + 1$ and $\varphi(s) = -1$.*

then for an arbitray $k \in (0,1)$

$$\begin{aligned} \int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)\, ds &= -\left(\frac{1}{M(Tx,Ty,t)} - 1\right) \\ &= -\left(\frac{1}{M(x,y,t)} - 1\right) \\ &\leq -k\left(\frac{1}{M(x,y,t)} - 1\right) \\ &= k\int_0^{\left(\frac{1}{M(x,y,t)}-1\right)} \varphi(s)\, ds; \end{aligned}$$

Thus (1) satisfied for all $k \in (0,1)$, but T has no fixed point.

Example 3.7. *Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ with fuzzy metric defined by $M(x,y,t) = \frac{1}{1+|x-y|}$. Define a map $T : X \to X$ by*

$$T(x) = \begin{cases} \frac{1}{n+1} & x = \frac{1}{n} n \in \mathbb{N} \\ 0 & x = 0 \end{cases}$$

then T is a integral fuzzy contraction with $\varphi(s) = s^{\frac{1}{s}-2}[1 - logs]$ and $k = \frac{1}{2}$.

Proof. Here

$$\int_0^u \varphi(s)du = u^{\frac{1}{u}}$$

Let $m, n \in \mathbb{N}$ with $n < m$ and let $x = \frac{1}{n}, y = \frac{1}{m}$, then we have

$nm < (n+1)(m+1)$ and $\frac{nm}{n-m} > 0$ so

$$\left(\frac{nm}{(n+1)(m+1)}\right)^{\frac{nm}{m-n}} \leq 1 \tag{7}$$

also $m \leq 3n + nm + 1$ or $2(m-n) \leq (n+1)(m+1)$ which implies that

$$\left(\frac{m-n}{(n+1)(m+1)}\right)^{\frac{n+m+1}{m-n}} \leq \frac{1}{2} \tag{8}$$

Now from (9) and (10) we get

$$\left(\frac{m-n}{(n+1)(m+1)}\right)^{\frac{n+m+1}{m-n}} \left(\frac{nm}{(n+1)(m+1)}\right)^{\frac{nm}{m-n}} \leq \frac{1}{2}$$

and

$$\left(\frac{m-n}{(n+1)(m+1)}\right)^{\frac{(n+1)(m+1)}{m-n}} \leq \frac{1}{2}\left(\frac{m-n}{nm}\right)^{\frac{nm}{m-n}}$$

$$\int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)ds \leq \frac{1}{2}\int_0^{\left(\frac{1}{M(x,y,t)}-1\right)} \varphi(s)ds \tag{9}$$

On the other hand when $x = \frac{1}{n}$ and $y = 0$ for $n \in \mathbb{N}$. Since $\frac{n}{n+1} \leq 1$ and $\frac{1}{n+1} \leq \frac{1}{2}$. Which implies that

$$\left[\frac{n}{n+1}\right]^n \frac{1}{n+1} \leq \frac{1}{2}$$

or

$$\left[\frac{1}{n+1}\right]^{n+1} \leq \frac{1}{2}\left[\frac{1}{n}\right]^n$$

which implies

$$\int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)ds \leq \int_0^{\left(\frac{1}{M(Tx,Ty,t)}-1\right)} \varphi(s)ds \tag{10}$$

Equation (9) and equation (10) shows that T is integral fuzzy contraction and has unique fixed point 0. But T is not fuzzy contraction because

$$\sup_{x,y\in X}\left(\frac{\frac{1}{M(Tx,Ty,t)}-1}{\frac{1}{M(x,y,t)}-1}\right) = 1$$

□

Acknowledgement: The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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