Res. Fixed Point Theory Appl.
Volume 2018, Article ID 2018007, 11 pages
eISSN 2581-6047

Results in Fixed Point Theory and Applications

RESEARCH ARTICLE

Fixed points of sequences of mappings

C. Ganesa Moorthy[1], **and S. Iruthaya Raj** *

1 Department of Mathematics, Alagappa University, Karaikudi-630 004, Tamil Nadu, India.

* PG and Research Department of Mathematics, Loyola College (Autonomous), Chennai-600034, Tamil Nadu, India.

Academic Editor: Fumiaki Kohsaka

Abstract: The concept of fixed points of a mapping is known. A new concept of fixed points of a sequence of mappings is introduced. Some results that are similar to the Banach contraction principle are obtained. A result for asymptotically non-expansive type mappings is also obtained.

Keywords: Fixed point, semi metric, Hausdorff metric.

MSC: 47H10.

1 Introduction

The Banach contraction principle confirms the existence of a fixed point of a contraction mapping from a complete metric space into itself. S. B. Nadler, Jr. [5] proves that a sequence of fixed points corresponding to a uniformly converging sequence of contraction mappings with a common Lipschitz constant converges to a fixed point of the limit function of the mappings on a complete metric space. This article [5] motivates to propose a definition of a fixed point of a sequence of mappings defined on a topological space. Let us recall that a fixed point of a mapping $f : X \longrightarrow X$ on a nonempty set X is a point x in X that satisfies the relation $f(x) = x$.

DOI: 10.30697/rfpta-2018-007
*Corresponding author E-mail: s.iruthayaraj@live.com(Iruthaya Raj)
Received May 22, 2018; Accepted October 13, 2018.

Definition 1.1. *Let $f_n : X \longrightarrow X, n = 1, 2, 3, \ldots$ be a sequence of mappings on a topological space X. A point $x \in X$ is said to be a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ of mappings, if $f_n(x)$ converges to x as $n \longrightarrow \infty$.*

If $f_n = f$ for every n, for some fixed f, then a point is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ when and only when it is a fixed point of the mapping f. Also, if f_n converges to f pointwisely, then a point is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ when and only when it is a fixed point of the mapping f. These facts justify the proposed definition 1.1. This definition requires some modifications for purposes of the present article.

A notation to be used in this article is $(X, (d_i)_{i \in I})$. This means that a nonempty set X is endowed with a family of semi metrics $d_i, i \in I$ on X (i.e., $d_i(x, y) = 0$ need not imply $x = y$) such that the topology induced by the subbase of all open balls generated by all d_i is Hausdorff. Since X endowed with this topology is a Tychonoff space, $(X, (d_i)_{i \in I})$ will be called as a Tychonoff space in this article. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to be Cauchy in $(X, (d_i)_{i \in I})$, if $d_i(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, for every $i \in I$. Let us say that a sequence $(x_n)_{n=1}^{\infty}$ in X converges to a point x in $(X, (d_i)_{i \in I})$, if $d_i(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$. If every Cauchy sequence in $(X, (d_i)_{i \in I})$ converges to some point in $(X, (d_i)_{i \in I})$, then let us say that $(X, (d_i)_{i \in I})$ is sequentially complete. A subset A of $(X, (d_i)_{i \in I})$ will be called as a sequentially complete subset of $(X, (d_i)_{i \in I})$, if $(A, (d_i)_{i \in I})$ is sequentially complete for d_i restricted to A. Let us modify the definition 1.1, for a Tychonoff space and this is to be followed in the present article.

Definition 1.2. *Let $(X, (d_i)_{i \in I})$ be a Tychonoff space. A sequence of mappings $f_n : X \longrightarrow X, n = 1, 2, 3, \ldots$ is said to have a fixed point x in X, if $(f_n(x))_{n=1}^{\infty}$ converges to x in $(X, (d_i)_{i \in I})$; that is, if $d_i(f_n(x), x) \longrightarrow 0$, as $n \longrightarrow \infty$, for every $i \in I$.*

The concepts and results of the present article are based on $(d_i)_{i \in I}$ and not only on the topology induced by $(d_i)_{i \in I}$. Some fixed point results are obtained for sequences of "contractions" and " asymptotically nonexpansive mappings". In the next Section 2, results are obtained for "contractions" on metric spaces. In Section 3, results are obtained for "asymptotically nonexpansive mappings". Uniformly convex Banach spaces with their modulii of convexity are considered in Section 3. For definitions and properties of these concepts, let us refer to the book [1].

Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space. Let $T : K \longrightarrow K$ satisfies $||T^n(x) - T^n(y)|| \leq k_n||x - y||, \forall x, y \in K$, for every $n = 1, 2, ...$, where $k_n > 1$ are numbers such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Then the fixed point set of T is nonempty closed and convex. This is a result obtained by K. Goebel and W. A. Kirk [2]. G. B. Passty [6] considers a sequence of mappings $T_n : K \longrightarrow K$ that satisfy $||T_n(x) - T_n(y)|| \leq k_n||x - y||, \forall x, y \in K$, for every n, where $k_n > 1$ and $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. This is further considered in Section 3.

2 Contractions

Theorem 2.1. *Let $(X, (d_i)_{i \in I})$ be a sequentially complete Tychonoff space. To each $i \in I$, let $(\alpha_{ni})_{n=1}^{\infty}$ be a sequence of positive reals such that $\sum_{n=1}^{\infty} (\alpha_{1i}\, \alpha_{2i} \ldots \alpha_{ni})$ converges. To each $i \in I$, let $(\beta_{ni})_{n=1}^{\infty}$ be a sequence of nonnegative reals such that $\sum_{n=1}^{\infty} \gamma_{ni}$ converges, when $\gamma_{ni} = \beta_{ni} + \alpha_{ni}\beta_{(n-1)i} + \alpha_{ni}\alpha_{(n-1)i}\beta_{(n-2)i} + \cdots + \alpha_{ni}\alpha_{(n-1)i} \cdots \alpha_{2i}\beta_{1i}$. Suppose further that, for each $i \in I$, there is a constant M_i such that $d_i(x, y) \leq M_i, \forall x, y \in X$. Let $f_n : X \longrightarrow X, n = 1, 2, 3, \ldots$ be a sequence of mappings such that*

$$d_i(f_n(x), f_{n+1}(y)) \quad \leq \quad \alpha_{ni} d_i(f_{n-1}(x), f_n(y)) + \beta_{ni} \tag{1}$$

$\forall n = 1, 2, 3, \ldots$ (with f_0 as identity mapping, or $\forall n = 2, 3, \ldots$), $\forall x, y \in X, \forall i \in I$. For given $x_0 \in X$, let a sequence $x_1, x_2, x_3, \ldots$ be defined by $x_n = f_n(x_{n-1})$. Then the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point x^* of the sequence $(f_n)_{n=1}^{\infty}$. Also, for every $i \in I$, we have

$$d_i(x^*, x_m) \quad \leq \quad M_i \left(\sum_{k=m}^{\infty} \alpha_{1i}\alpha_{2i} \cdots \alpha_{ki} \right) + \sum_{k=m}^{\infty} \gamma_{ki}. \tag{2}$$

Moreover x^* is the unique fixed point of the sequence $(f_n)_{n=1}^{\infty}$.

Proof. Note that

$$\begin{aligned} d_i(x_{n+1}, x_n) &= d_i(f_{n+1}(x_n), f_n(x_{n-1})) \\ &\leq \alpha_{ni} d_i(f_n(x_n), f_{n-1}(x_{n-1})) + \beta_{ni} \\ &\leq \alpha_{ni}\alpha_{(n-1)i} d_i(f_{n-1}(x_n), f_{n-2}(x_{n-1})) + \beta_{ni} + \alpha_{ni}\beta_{(n-1)i} \\ &\leq \alpha_{ni}\alpha_{(n-1)i} \cdots \alpha_{2i}\alpha_{1i} d_i(f_1(x_n), x_{n-1}) + \gamma_{ni} \\ &\leq M_i \alpha_{1i}\alpha_{2i} \cdots \alpha_{ni} + \gamma_{ni}. \end{aligned}$$

Therefore, for $n > m \geq 1$, we have

$$\begin{aligned} d_i(x_n, x_m) &\leq d_i(x_m, x_{m+1}) + d_i(x_{m+1}, x_{m+2}) + \cdots + d_i(x_{n-1}, x_n) \\ &\leq M_i \left(\sum_{k=m}^{n-1} \alpha_{1i}\alpha_{2i} \cdots \alpha_{ki} \right) + \sum_{k=m}^{n-1} \gamma_{ki}. \end{aligned} \tag{3}$$

So, by our assumption on $(\alpha_{ni})_{n=1}^{\infty}$ and $(\beta_{ni})_{n=1}^{\infty}$, the sequence $(x_n)_{n=1}^{\infty}$ becomes a Cauchy sequence. Let $(x_n)_{n=1}^{\infty}$ converge to x^* in X. Now,

$$\begin{aligned} d_i(x^*, f_n(x^*)) &\leq d_i(x^*, x_{n+1}) + d_i(x_{n+1}, f_n(x^*)) \\ &= d_i(x^*, x_{n+1}) + d_i(f_{n+1}(x_n), f_n(x^*)) \\ &\leq d_i(x^*, x_{n+1}) + \alpha_{ni} d_i(f_n(x_n), f_{n-1}(x^*)) + \beta_{ni} \\ &\leq d_i(x^*, x_{n+1}) + \alpha_{ni}\alpha_{(n-1)i} \cdots \alpha_{1i} d_i(f_1(x_n), x^*) + \gamma_{ni} \\ &\leq d_i(x^*, x_{n+1}) + M_i \alpha_{1i}\alpha_{2i} \cdots \alpha_{ni} + \gamma_{ni}. \end{aligned}$$

Thus $(f_n(x^*))_{n=1}^{\infty}$ converges to x^* as $n \longrightarrow \infty$, because $\alpha_{1i}\alpha_{2i} \cdots \alpha_{ni} \longrightarrow 0$ and $\gamma_{ni} \longrightarrow 0$ as $n \longrightarrow \infty$. Note that (2) follows from (3).

If x and y are two fixed points of the sequence $(f_n)_{n=1}^{\infty}$, then

$$\begin{aligned} 0 &\leq d_i(x,y) \\ &\leq d_i(x, f_n(x)) + d_i(f_n(x), f_{n+1}(y)) + d_i(f_{n+1}(y), y) \\ &\leq d_i(x, f_n(x)) + \alpha_{ni}\alpha_{(n-1)i}\cdots\alpha_{1i}M_i + \gamma_{ni} + d_i(f_{n+1}(y), y), \end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$. Thus $d_i(x,y) = 0$, for all $i \in I$ and hence $x = y$. This proves the uniqueness of the fixed point of $(f_n)_{n=1}^{\infty}$. □

A strategy of the article [4] is applied now to derive the following extension.

Theorem 2.2. *Let $(X,(d_i)_{i\in I})$, $(\alpha_{ni})_{n=1}^{\infty}, i \in I$, $(\beta_{ni})_{n=1}^{\infty}, i \in I$ and $M_i, i \in I$ be as in the theorem 2.1. Let $A_1, A_2, \dots, A_p$ be a finite number of nonempty sequentially complete subsets of X such that $X = A_1 \cup A_2 \cup \dots \cup A_p$. Let $f_n : X \longrightarrow X, n = 1,2,3,\dots$ be a sequence of mappings such that to each n, we have $f_n(A_k) \subseteq A_{k+1}$ for every $k = 1,2,\dots,p$, with $A_{p+1} = A_1$. Suppose further that (1) holds for $n = 2,3,\dots$, for $x \in A_k$ and $y \in A_{k+1}, k = 1,2,\dots,p$ and for $i \in I$. For given $x_0 \in X$, let a sequence $x_1, x_2, \dots$ be defined by $x_n = f_n(x_{n-1})$. Then the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point x^* of the sequence $(f_n)_{n=1}^{\infty}$. Also, this fixed point is unique in X.*

Proof. The arguments of the proof of the previous theorem 2.1 imply that the sequence $(x_n)_{n=1}^{\infty}$ converges to a point x^* of the sequence $(f_n)_{n=1}^{\infty}$. Since each A_k contains a subsequence of $(x_n)_{n=1}^{\infty}$ and since each A_k is sequentially complete, $x^* \in A_k$ for every k. So, $x^* \in \bigcap_{k=1}^{p} A_k$ so that $\bigcap_{k=1}^{p} A_k$ is nonempty. Now again the arguments used in the proof of the previous theorem show that x^* is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$.

If $(f_n(x))_{n=1}^{\infty}$ converges to x, for some $x \in X$, then each A_k contains a subsequence of $(f_n(x))_{n=1}^{\infty}$ so that $x \in \bigcap_{k=1}^{p} A_k$. Now, uniqueness of the fixed point x^* in X follows from the uniqueness part of the previous theorem 2.1 applied to $\bigcap_{k=1}^{p} A_k$. □

Consider a Tychonoff space $(X,(d_i)_{i\in I})$. To each subset A of X, an $i \in I$ and a $x \in X$, let $d_i(x,A)$ or $d_i(A,x)$ denote the number $\inf\{d_i(x,y) : y \in A\}$. Let us say that a subset A of X is closed in $(X,(d_i)_{i\in I})$ if $d_i(x,A) = 0, \forall i \in I$ implies $x \in A$. Let us say that a subset A of X is bounded in $(X,(d_i)_{i\in I})$, if to each $i \in I$, there is a positive number M_i such that $d_i(x,y) \leq M_i, \forall x,y \in A$. Let $CB(X)$ denote the collection of all nonempty closed bounded subsets of $(X,(d_i)_{i\in I})$. To each $i \in I$ and each $A, B \in CB(X)$, let $H_i(A,B) = \sup\left\{\sup_{x\in A} d_i(x,B), \sup_{y\in B} d_i(A,y)\right\}$. Then H_i is a Hausdorff semi metric on $CB(X)$. Moreover, $(CB(X),(H_i)_{i\in I})$ is a Tychonoff space (see Example 2.15 in [1] for arguments). With these notations, let us define fixed points for sequences of set valued mappings.

Definition 2.3. *Let $f_n : X \longrightarrow CB(X), n = 1,2,\dots$ be a sequence of set valued mappings. A point $x \in X$ is defined as a fixed point of the sequence $(f_n)_{n=1}^{\infty}$, if $d_i(f_n(x),x) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$.*

Although the definition 2.3 is stated for a general Tychonoff space, the next result is obtained only for a metric space.

Theorem 2.4. *Let (X,d) be a bounded complete metric space. Let H be the Hausdorff metric defined on $CB(X)$ by d. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_1\alpha_2\cdots\alpha_n < \infty$. Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $\sum_{n=2}^{\infty} \gamma_n$ converges, when $\gamma_n = \beta_n + \alpha_n\beta_{n-1} + \alpha_n\alpha_{n-1}\beta_{n-2} + \cdots + \alpha_n\alpha_{n-1}\cdots\alpha_3\beta_2$. Let $f_n : X \longrightarrow CB(X), n = 1,2,\dots$ be a sequence of mappings such that*

$$H(f_n(x), f_{n+1}(y)) \leq \alpha_n H(f_{n-1}(x), f_n(y)) + \beta_n \tag{4}$$

$\forall n = 2,3,\dots$, and for all $x,y \in X$. Then there is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ in X.

Proof. Suppose $d(x,y) \leq M, \forall x,y \in X$, for some $M > 0$. Fix $x_0 \in X$. Find points $x_1, x_2, \dots$ in X such that $x_n \in f_n(x_{n-1})$, for $n = 1,2,\dots$. Now

$$\begin{aligned} H(f_{n+1}(x_n), f_n(x_{n-1})) &\leq \alpha_n H(f_n(x_n), f_{n-1}(x_{n-1})) + \beta_n \\ &\leq \alpha_n\alpha_{n-1}\cdots\alpha_2 H(f_2(x_n), f_1(x_{n-1})) + \gamma_n \\ &\leq M\alpha_2\alpha_3\cdots\alpha_n + \gamma_n. \end{aligned}$$

Fix $y_1 \in f_1(x_0), y_2 \in f_2(x_1), y_3 \in f_3(x_2), \dots$ such that

$$d(y_n, y_{n+1}) \leq H(f_n(x_{n-1}), f_{n+1}(x_n)) + \frac{1}{2^n}, \text{ for } n \geq 1.$$

Now, for $n > m \geq 2$, we have

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m) \\ &\leq H(f_n(x_{n-1}), f_{n-1}(x_{n-2})) + H(f_{n-1}(x_{n-2}), f_{n-2}(x_{n-3})) \\ &\quad + \cdots + H(f_{m+1}(x_m), f_m(x_{m-1})) + \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} \\ &\leq M\sum_{k=m}^{n-1} \alpha_2\alpha_3\cdots\alpha_k + \sum_{k=m}^{n-1}\frac{1}{2^k} + \sum_{k=m}^{n-1}\gamma_k. \end{aligned}$$

By our assumption on $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$, the sequence $(y_n)_{n=1}^{\infty}$ converges to some y^* in (X,d).

For any $z_n \in f_n(y^*)$ and $u_{n+1} \in f_{n+1}(x_n)$, we have

$$\begin{aligned} d(f_n(y^*), y^*) &\leq d(z_n, y^*) \\ &\leq d(z_n, u_{n+1}) + d(u_{n+1}, y_{n+1}) + d(y_{n+1}, y^*). \\ \text{So, } d(f_n(y^*), y^*) &\leq d(f_n(y^*), u_{n+1}) + d(u_{n+1}, y_{n+1}) + d(y_{n+1}, y^*) \\ &\leq H(f_n(y^*), f_{n+1}(x_n)) + d(u_{n+1}, y_{n+1}) + d(y_{n+1}, y^*). \\ \text{So, } d(f_n(y^*), y^*) &\leq H(f_n(y^*), f_{n+1}(x_n)) + d(f_{n+1}(x_n), y_{n+1}) + d(y_{n+1}, y^*) \\ &= H(f_n(y^*), f_{n+1}(x_n)) + d(y_{n+1}, y^*) \\ &\leq M\alpha_n\alpha_{n-1}\cdots\alpha_2 + \gamma_n + d(y_{n+1}, y^*). \end{aligned}$$

Since $\alpha_1\alpha_2\cdots\alpha_n \longrightarrow 0$, $\gamma_n \longrightarrow 0$ and $y_{n+1} \longrightarrow y^*$ as $n \longrightarrow \infty$, $d(f_n(y^*), y^*) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus y^* is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$. □

Theorem 2.5. *Let (X,d), H, $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ be as in the previous theorem 2.4. Let $A_1, A_2, \ldots, A_p$ be a finite number of nonempty closed subsets of X such that $X = A_1 \cup A_2 \cup \cdots \cup A_p$. Let $f_n : X \longrightarrow CB(X), n = 1,2,\ldots$ be a sequence of mappings such that $f_n(x) \subseteq A_{k+1}$ whenever $x \in A_k, \forall k = 1,2,\ldots,p$, with $A_{p+1} = A_1$. Suppose further that (4) holds for $n = 2,3,\ldots,$ for $x \in A_k$ and $y \in A_{k+1}$, $k = 1,2,\ldots,p$. Then there is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ in X.*

Proof. Suppose $d(x,y) \leq M, \forall x,y \in X$, for some $M > 0$. Find points $x_1, x_2, \ldots$ in X such that $x_n \in f_n(x_{n-1})$, for $n = 1,2,\ldots$. Fix $y_1 \in f_1(x_0)$, $y_2 \in f_2(x_1)$, $y_3 \in f_3(x_2)\ldots$, such that $d(y_n, y_{n+1}) \leq H(f_n(x_{n-1}), f_{n+1}(x_n)) + \frac{1}{2^n}$, for $n \geq 1$. Now the proof of the previous theorem 2.4 implies that $(y_n)_{n=1}^{\infty}$ converges to some y^* in X so that $y^* \in \bigcap_{k=1}^{p} A_k$. Note that $f_n(x) \subseteq \bigcap_{k=1}^{p} A_k$, for every $x \in \bigcap_{k=1}^{p} A_k$ and $\forall n \geq 1$. Now the previous theorem 2.4 can be applied to prove the existence of a fixed point of $(f_n)_{n=1}^{\infty}$, when it is applied to $\bigcap_{k=1}^{p} A_k$. Also, the proof of the previous theorem implies that y^* is a fixed point of $(f_n)_{n=1}^{\infty}$ in X.□

Theorem 2.6. *Let $(X, (d_i)_{i\in I})$ and $(M_i)_{i \in I}$ be as in theorem 2.1. To each $i \in I$, let $\alpha_i \in (0,1)$. Let k be a fixed positive integer. Let $f_n : X \longrightarrow X, n = 1,2,\ldots$ be a sequence of mappings such that*

$$\begin{aligned} d_i(f_{n+1}(x), f_n(y)) &\leq \alpha_i \max\{d_i(f_n(x), f_{n-1}(y)), d_i(f_{n-1}(x), f_{n-2}(y)), \\ &\quad \ldots, d_i(f_{n-k+1}(x), f_{n-k}(y))\} \end{aligned} \tag{5}$$

for $n \geq k+1$. For given $x_0 \in X$, let a sequence $x_1, x_2, \ldots$ be defined by $x_n = f_n(x_{n-1}), n = 1,2,3,\ldots$. Then the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point x^* of the sequence $(f_n)_{n=1}^{\infty}$. Also, for every $i \in I$, we have

$$d_i(x^*, x_m) \leq M_i \sum_{r=m}^{\infty} \alpha_i^{\frac{r-1}{k}-1} \tag{6}$$

for $m \geq k+1$. Moreover x^* is the unique fixed point of the sequence $(f_n)_{n=1}^{\infty}$.

Proof. Note that for $n \geq k+1$ we have

$$\begin{aligned} d_i(x_{n+1}, x_n) &= d_i(f_{n+1}(x_n), f_n(x_{n-1})) \\ &\leq \alpha_i \max\{d_i(f_n(x_n), f_{n-1}(x_{n-1})), d_i(f_{n-1}(x_n), f_{n-2}(x_{n-1})), \\ &\quad \ldots, d_i(f_{n-k+1}(x_n), f_{n-k}(x_{n-1}))\} \\ &\leq \alpha_i^{[\frac{n-1}{k}]} M_i, \end{aligned}$$

where $[\frac{n-1}{k}]$ is the integer part of $\frac{n-1}{k}$ and when the last inequality is derived by successive application of (5) with a notification that maximum of a finite set is reached at some member of the set. Therefore, for $n > m \geq k+1$, we have

$$d_i(x_n, x_m) \leq M_i \sum_{r=m}^{n-1} \alpha_i^{\frac{r-1}{k}-1}. \tag{7}$$

Since $\alpha_i \in (0,1)$, the sequence $(x_n)_{n=1}^{\infty}$ becomes a Cauchy sequence. Let $(x_n)_{n=1}^{\infty}$ converge to $x^* \in X$. Now,

$$\begin{aligned} d_i(x^*, f_n(x^*)) &\leq d_i(x^*, x_{n+1}) + d_i(f_{n+1}(x_n), f_n(x^*)) \\ &\leq d_i(x^*, x_{n+1}) + M_i \alpha_i^{\frac{n-1}{k}-1}. \end{aligned}$$

Thus $f_n(x^*) \longrightarrow x^*$ as $n \longrightarrow \infty$. Now (7) implies (6). Also, with $n \geq k+1$,

$$\begin{aligned} 0 \leq d_i(x,y) &\leq d_i(x, f_n(x)) + d_i(f_n(x), f_{n+1}(y)) + d_i(f_{n+1}(y), y) \\ &\leq d_i(x, f_n(x)) + M_i \alpha_i^{\frac{n-1}{k}-1} + d_i(f_{n+1}(y), y) \end{aligned}$$

implies the uniqueness of the fixed point of $(f_n)_{n=1}^{\infty}$. □

Theorem 2.7. *Let $(X,(d_i)_{i\in I})$, $(\alpha_i)_{i\in I}$ and $(M_i)_{i\in I}$ be as in theorem 2.6. Let $A_1, A_2, \ldots, A_p$ and $f_n : X \longrightarrow X, n = 1,2,\ldots$ be as in theorem 2.2. Suppose further that (5) holds for some k, for $n = k+1, k+2, \ldots$, for $x \in A_{r+1}$ and $y \in A_r$, $r = 1,2,\ldots,p$ and for $i \in I$ (with $A_{p+1} = A_1$). For given $x_0 \in X$, let a sequence $x_1, x_2, \ldots$ be defined by $x_n = f_n(x_{n-1})$, $n = 1,2,\ldots$. Then the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point of the sequence $(f_n)_{n=1}^{\infty}$. Also, this fixed point is unique in X.*

Proof. The arguments of the proofs of the theorem 2.2 and the theorem 2.6 are applicable. □

The theorems 2.6 and 2.7 correspond to theorems 2.1 and 2.2 for a new type of mappings. The following theorems 2.8 and 2.9 correspond to theorems 2.4 and 2.5. One can modify the arguments to get required proofs.

Theorem 2.8. *Let (X,d), H and $CB(X)$ be as in theorem 2.4. Let $\alpha \in (0,1)$ be a given number. Let k be a fixed positive integer. Let $f_n : X \longrightarrow CB(X)$, $n = 1,2,\ldots$ be a sequence of mappings such that for $n \geq k+1$, we have*

$$\begin{aligned} H(f_n(x), f_{n+1}(y)) &\leq \alpha \max\{H(f_{n-1}(x), f_n(y)), H(f_{n-2}(x), f_{n-1}(y)), \\ &\qquad \ldots, H(f_{n-k}(x), f_{n-k+1}(y))\} \end{aligned} \tag{8}$$

for all $n = k+1, k+2, , \ldots$ and for all $x, y \in X$. Then $(f_n)_{n=1}^{\infty}$ has a fixed point in X.

Theorem 2.9. *Let (X,d), H, $CB(X)$ and α be as in theorem 2.8. Let $A_1, A_2, \ldots, A_p$ be a finite number of nonempty closed subsets of (X,d) such that $X = A_1 \cup A_2 \cup \cdots \cup A_p$. Let $f_n : X \longrightarrow CB(X)$, $n = 1,2,\ldots$ be a sequence of mappings such that $f_n(x) \subseteq A_{r+1}$ whenever $x \in A_r$, $\forall r = 1,2,\ldots,p$, when $A_{p+1} = A_1$. Let k be a positive integer. Suppose further that (8) holds for $n = k+1, k+2, \ldots$, for $x \in A_r$ and $y \in A_{r+1}$, $r = 1,2,\ldots,p$. Then there is a fixed point of the sequence $(f_n)_{n=1}^{\infty}$ in X.*

Theorem 2.10. *Let $(X,(d_i)_{i\in I})$, $(\alpha_{ni})_{n=1}^{\infty}$, $i \in I$, $(\beta_{ni})_{n=1}^{\infty}$, $i \in I$ and $(\gamma_{ni})_{n=1}^{\infty}$, $i \in I$ be as in the theorem 2.1. Suppose $(\alpha_{ni})_{n=1}^{\infty}$ is bounded and $(\beta_{ni})_{n=1}^{\infty}$ converges to 0, for every $i \in I$. Let $f_n : X \longrightarrow X$, $n = 1,2,\ldots$ be a sequence of mappings such that*

$$d_i(f_n(x), f_{n+1}(y)) \leq \alpha_{ni} d_i(x,y) + \beta_{ni} \tag{9}$$

$\forall i \in I$, $\forall x, y \in X$, $\forall n = 1,2,\ldots$. For given $x_0 \in X$, let a sequence $x_1, x_2, \ldots$ be defined by $x_n = f_n(x_{n-1})$. Then the sequence $(x_n)_{n=1}^{\infty}$ converges to a fixed point x^* of the sequence $(f_n)_{n=1}^{\infty}$. Also, for every $i \in I$,

$$d_i(x^*, x_m) \leq \left(\sum_{k=m}^{\infty} \alpha_{1i}\alpha_{2i}\cdots\alpha_{ki} \right) d_i(x_0, x_1) + \sum_{k=m}^{\infty} \gamma_{ki}. \tag{10}$$

Moreover, if $y_0 \in X$, $y_n = f(y_{n-1})$, $n = 1,2,\ldots$ and $d_i(y_n, y^*) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$, for some $y^* \in X$, then $x^* = y^*$.

Proof. Note that

$$\begin{aligned} d_i(x_{n+1}, x_n) &= d_i(f_{n+1}(x_n), f_n(x_{n-1})) \\ &\leq \alpha_{ni} d_i(x_n, x_{n-1}) + \beta_{ni} \\ &\leq \alpha_{ni}\alpha_{(n-1)i}\cdots\alpha_{2i}\alpha_{1i} d_i(x_1, x_0) + \gamma_{ni} \end{aligned}$$

Therefore, for $n > m \geq 1$, we have

$$\begin{aligned} d_i(x_n, x_m) &\leq d_i(x_m, x_{m+1}) + d_i(x_{m+1}, x_{m+2}) + \cdots + d_i(x_{n-1}, x_n) \\ &\leq \left(\sum_{k=m}^{n-1} \alpha_{1i}\alpha_{2i}\cdots\alpha_{ki}\right) d_i(x_0, x_1) + \sum_{k=m}^{n-1} \gamma_k. \end{aligned} \tag{11}$$

So, by our assumption on $(\alpha_{ni})_{n=1}^{\infty}$ and $(\beta_{ni})_{n=1}^{\infty}$, the sequence $(x_n)_{n=1}^{\infty}$ becomes a Cauchy sequence. Let $(x_n)_{n=1}^{\infty}$ converge to $x^* \in X$. Then we have

$$\begin{aligned} d_i(x^*, f_n(x^*)) &\leq d_i(x^*, x_{n+1}) + d_i(x_{n+1}, f_n(x^*)) \\ &= d_i(x^*, x_{n+1}) + d_i(f_{n+1}(x_n), f_n(x^*)) \\ &\leq d_i(x^*, x_{n+1}) + \alpha_{ni} d_i(x_n, x^*) + \beta_{ni}. \end{aligned}$$

Thus $d_i(x^*, f_n(x^*)) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$, because $(\alpha_{ni})_{n=1}^{\infty}$ is a bounded sequence and $(\beta_{ni})_{n=1}^{\infty}$ converges to zero, for every $i \in I$. Note that (10) follows from (11).

When $y_0 \in X$ and $y_n = f_n(y_{n-1})$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} d_i(x_n, y_{n+1}) &= d_i(f_n(x_{n-1}), f_{n+1}(y_n)) \\ &\leq \alpha_{ni} d_i(x_{n-1}, y_n) + \beta_{ni} \\ &\leq \alpha_{ni}\alpha_{(n-1)i}\cdots\alpha_{2i}\alpha_{1i} d_i(x_0, y_1) + \gamma_{ni}. \end{aligned}$$

So, if $d_i(y_n, y^*) \longrightarrow 0$ as $n \longrightarrow \infty$, $\forall i \in I$, for some $y^* \in X$, then we have $x^* = y^*$, because $\alpha_{1i}\alpha_{2i}\cdots\alpha_{ni} \longrightarrow 0$ and $\gamma_{ni} \longrightarrow 0$ as $n \to \infty$, $\forall i \in I$. This proves the theorem. □

Lemma 2.11. *Let $(X, (d_i)_{i\in I})$ be a Tychonoff space. To each $i \in I$, let $(\beta_{ni})_{n=1}^{\infty}$ be a sequence of nonnegative numbers that converges to 0 as $n \longrightarrow \infty$. To each $i \in I$, let $(\alpha_{ni})_{n=1}^{\infty}$ be a sequence of positive numbers such that $\limsup_{n\longrightarrow\infty} \alpha_{ni} < 1$. Suppose $(f_n)_{n=1}^{\infty}$ be as in the previous theorem 2.10. Then the sequence $(f_n)_{n=1}^{\infty}$ can have at most one fixed point if it exists.*

Proof. Suppose $x, y \in X$ be such that $d_i(f_n(x), x) \longrightarrow 0$ and $d_i(f_n(y), y) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$. Then

$$\begin{aligned} d_i(x, y) &\leq d_i(x, f_n(x)) + d_i(f_n(x), f_{n+1}(y)) + d_i(f_{n+1}(y), y) \\ &\leq d_i(x, f_n(x)) + \alpha_{ni} d_i(x, y) + \beta_{ni} + d_i(f_{n+1}(y), y). \end{aligned}$$

So, if $x \neq y$, then the assumption on α_{ni} and β_{ni} would imply $d_i(x, y) < d_i(x, y)$ for every $i \in I$ for which $d_i(x, y) \neq 0$. This proves that $x = y$. □

Corollary 2.12. *Let $(X,(d_i)_{i\in I})$, $(\alpha_{ni})_{n=1}^{\infty}$, $i \in I$, $(\beta_{ni})_{n=1}^{\infty}$ and f_n for $n = 1, 2, \dots$ be as in theorem 2.10. Assume that $\beta_{ni} \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$. Suppose further that $(\alpha_{ni})_{n=1}^{\infty}$ is a sequence such that $\limsup_{n\longrightarrow\infty} \alpha_{ni} < 1$, for every $i \in I$. Then the sequence $(f_n)_{n=1}^{\infty}$ has a unique fixed point in X.*

Proof. The proof follows from theorem 2.10 and lemma 2.11 □

If $y = x$ and if $\beta_{ni} = 0$, for every n and for every i, then (9) implies that $f_n(x) = f_{n+1}(x)$ and so theorem 2.10 provides a result for a single function. The following result for a single function based on a strategy of the article [4] seems to be new.

Theorem 2.13. *Let (X, d) be a complete metric space. Let H be the Hausdorff metric defined on $CB(X)$. Let $A_1, A_2, \dots, A_p$ be a finite number of nonempty closed subsets of (X, d) such that $X = \bigcup_{k=1}^{p} A_k$. Let $f : X \longrightarrow CB(X)$ be a mapping such that $f(x) \subseteq A_{k+1}$ for $x \in A_k$, $k = 1, 2, \dots, p$, with $A_{p+1} = A_1$. Suppose $H(f(x), f(y)) \leq \alpha d(x, y)$, for all $x \in A_k$, for all $y \in A_{k+1}$, $k = 1, 2, \dots, p$ and for some $\alpha \in (0, 1)$. Then f has a fixed point x^* in X. That is $x^* \in f(x^*)$.*

Proof. Fix $x_0 \in X$ and $x_1 \in f(x_0)$. Then we find $x_{n+1} \in f(x_n)$, $n = 1, 2, \dots$, successively such that $d(x_n, x_{n+1}) \leq H(f(x_{n-1}), f(x_n)) + \alpha^n$, $n = 1, 2, \dots$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(f(x_{n-1}), f(x_n)) + \alpha^n \\ &\leq \alpha d(x_{n-1}, x_n) + \alpha^n \\ &\leq \alpha[H(f(x_{n-2}), f(x_{n-1})) + \alpha^{n-1}] + \alpha^n \\ &\leq \alpha^2 d(x_{n-2}, x_{n-1}) + 2\alpha^n \\ &\leq \alpha^n d(x_0, x_1) + n\alpha^n. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq d(x_0, x_1) \left(\sum_{n=1}^{\infty} \alpha^n \right) + \sum_{n=1}^{\infty} n\alpha^n < \infty$. Thus $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $(x_n)_{n=1}^{\infty}$ converge to x^*. Since each A_k contains a subsequence of $(x_n)_{n=1}^{\infty}$ and A_k is closed, $x^* \in A_k$, $\forall k = 1, 2, \dots, p$. Thus $\bigcap_{k=1}^{p} A_k \neq \emptyset$ and $f(x) \subseteq \bigcap_{k=1}^{p} A_k$, $\forall x \in \bigcap_{k=1}^{p} A_k$. So, this theorem follows from the corresponding classical result (Theorem 3.20 in [3]), when f is restricted to $\bigcap_{k=1}^{p} A_k$. □

3 Asymptotically Nonexpansive Mappings

The arguments of K. Goebel and W. A. Kirk [2] (see also Chapter 5 in [1]) are modified to obtain the following fixed point result.

Theorem 3.1. *Let K be a nonempty closed convex and bounded subset of a uniformly convex Banach space $(X, ||\ ||)$. Let $(k_n)_{n=1}^{\infty}$ be a sequence of numbers greater than 1 such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of nonnegative numbers that converges to 0. Let $T_n : K \longrightarrow K$, $n = 1, 2, \dots$ be mappings that satisfy $||T_{n+i}(x) - T_n(y)|| \leq k_n ||T_i(x) - y|| + \beta_n$, for all positive integers i and n and for all $x, y \in K$. Then there is a fixed point of the sequence $(T_n)_{n=1}^{\infty}$. Moreover, the set of all fixed points of $(T_n)_{n=1}^{\infty}$ is a closed subset of K.*

Proof. For each $x \in K$ and $r > 0$, let $B(x,r) = \{y \in X : ||x-y|| \leq r\}$. Let $y \in K$ be fixed. Let ρ_0 be the infimum of all $\rho > 0$ such that $K \bigcap (\bigcap_{i=k}^{\infty} B(T_i(y), \rho)) \neq \emptyset$, for some k that depends on ρ. To each $\epsilon > 0$, let $C_\epsilon = \bigcup_{k=1}^{\infty} (\bigcap_{i=k}^{\infty} B(T_i(y), \rho_0 + \epsilon))$. Then $C_\epsilon \bigcap K$ are nonempty and convex. Let $C = \bigcap_{\epsilon > 0} (\overline{C_\epsilon} \bigcap K)$, where $\overline{C_\epsilon}$ is the closure of C_ϵ. Then $C \neq \emptyset$, because K is weakly compact. Note that for $x \in C$ and $\eta > 0$, there exists an integer n such that if $i \geq n$, then $||x - T_i(y)|| \leq \rho_0 + \eta$.

Let $x \in C$ be fixed. Suppose $(T_n(x))_{n=1}^{\infty}$ does not converge to x. Then there exists $\epsilon > 0$ such that for every given positive integer n, there is another integer $m > n$ such that $||T_m(x) - x|| \geq \epsilon$. For $m > n$, we have $||T_n(x) - T_m(x)|| \leq k_n ||x - T_{m-n}(x)|| + \beta_n$. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that $\left(1 - \delta\left(\frac{\epsilon}{\rho_0 + \alpha}\right)\right)(\rho_0 + \alpha) < \rho_0$, where δ is the modulus of convexity of the given uniformly convex Banach space. Select n so that $||x - T_n(x)|| \geq \epsilon$ and also so that $k_n\left(\rho_0 + \frac{\alpha}{2}\right) + \beta_n \leq \rho_0 + \alpha$. There is an integer $n_0 \geq n$ such that $m > n_0$ implies $||x - T_{m-n}(y)|| \leq \rho_0 + \frac{\alpha}{2}$. Then we have $||T_n(x) - T_m(y)|| \leq k_n ||x - T_{m-n}(y)|| + \beta_n \leq \rho_0 + \alpha$ and $||x - T_m(y)|| \leq \rho_0 + \alpha$, for $m > n_0 (\geq n)$. Thus by uniform convexity of X, if $m > n_0$, we have $\left|\left|\frac{x + T_n(x)}{2} - T_m(y)\right|\right| \leq \left(1 - \delta\left(\frac{\epsilon}{\rho_0 + \alpha}\right)\right)(\rho_0 + \alpha) < \rho_0$. This contradicts the definition of ρ_0. Hence we conclude that $\rho_0 = 0$ or $T_m(x) \longrightarrow x$ as $m \longrightarrow \infty$.

Suppose $\rho_0 = 0$. Then $||T_k(y) - x|| \longrightarrow 0$ as $k \longrightarrow \infty$. So, $||T_{m+k}(y) - T_m(x)|| \leq k_m ||T_k(y) - x|| + \beta_m$ implies that $||T_{m+k}(y) - T_m(x)|| \longrightarrow 0$ as $k \longrightarrow \infty$ and $m \longrightarrow \infty$. Now, $0 \leq ||T_p(x) - x|| \leq ||T_p(x) - T_{p+m}(y)|| + ||T_{p+m}(y) - x||$ and the right hand side tends to zero as $p \longrightarrow \infty$ and $m \longrightarrow \infty$. Thus $T_p(x) \longrightarrow x$ as $p \longrightarrow \infty$. In any case, we have the conclusion $T_m(x) \longrightarrow x$ as $m \longrightarrow \infty$, for any fixed $x \in C$. Thus x is a fixed point of the sequence $(T_m)_{m=1}^{\infty}$, for any $x \in C \neq \emptyset$.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of fixed points of $(T_m)_{m=1}^{\infty}$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ to some x in K. Fix $\epsilon > 0$. Find an integer j such that $k_n ||x_j - x|| \leq \frac{\epsilon}{8}, \forall n$. Find an integer i_0 such that $k_n ||T_i(x_j) - x_j|| \leq \frac{\epsilon}{8}$, $\forall i \geq i_0, \forall n$. Then $k_n ||T_i(x_j) - x|| \leq \frac{\epsilon}{4}, \forall i \geq i_0, \forall n$. Find an integer n_0 such that $\beta_n \leq \frac{\epsilon}{2}, \forall n \geq n_0$. Then we have $||T_n(x) - x|| \leq ||T_{n+i}(x_j) - x|| + ||T_{n+i}(x_j) - T_n(x)|| \leq ||T_{n+i}(x_j) - x|| + k_n ||T_i(x_j) - x|| + \beta_n \leq \epsilon$, $\forall n \geq n_0, \forall i \geq i_0$. This proves that x is also a fixed point of $(T_m)_{m=1}^{\infty}$. So, the fixed points of $(T_m)_{m=1}^{\infty}$ form a closed subset of K. □

Corollary 3.2. *Let us assume all assumptions of the previous theorem* 3.1. *Suppose further that* $||T_i(x) - T_i(y)|| \leq k_i ||x - y|| + \beta_i, \forall x, y \in K, \forall i = 1, 2, \ldots$ *Then the fixed points of* $(T_n)_{n=1}^{\infty}$ *form a nonempty closed convex subset of* K.

Proof. Let δ denote the modulus of convexity of X. Consider two fixed points x and y of $(T_n)_{n=1}^{\infty}$. Write $z = \frac{x+y}{2}$. To complete the proof, it will be shown that z is also a fixed point of $(T_n)_{n=1}^{\infty}$. There is a sequence $(\epsilon_k)_{k=1}^{\infty}$ of positive numbers such that $k_i ||x - T_k(x)|| \leq \epsilon_k$ and $k_i ||y - T_k(y)|| \leq \epsilon_k$, $\forall i, \forall k$ and such that

$\epsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$. Then we have

$$\begin{aligned}
||T_i(z) - x|| &\leq ||T_i(z) - T_i(x)|| + ||T_i(x) - x|| \\
&\leq k_i||z - x|| + \beta_i + \epsilon_i \\
&= \frac{k_i}{2}||x - y|| + \beta_i + \epsilon_i, \text{ and} \\
||T_i(z) - y|| &\leq \frac{k_i}{2}||x - y|| + \beta_i + \epsilon_i. \text{ So,} \\
||z - T_i(z)|| &\leq \left(1 - \delta\left(\frac{||x-y||}{\frac{k_i}{2}||x-y|| + \beta_i + \epsilon_i}\right)\right)\left(\frac{k_i}{2}||x - y|| + \beta_i + \epsilon_i\right).
\end{aligned}$$

Thus $||z - T_i(z)|| \longrightarrow 0$ as $i \longrightarrow \infty$ and the corollary is proved. □

References

[1] R. P. Agarval, D. O'Regan and D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications, Springer, New York, 2009.

[2] K. G. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., 35(1972) 171-174.

[3] M. A. Khamsi and W. A. Kirk, An introduction to metric spaces and fixed point theory, John-Wiley, New York, 2001.

[4] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed point theory, 4 (2003), 79-89.

[5] Sam B. Nadler, Jr., Sequences of contractions and fixed points, Pacific Journal of Mathematics, 27 (No. 3) (1968), 579-585.

[6] G. B. Passty, Construction of fixed points for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., 84(1982) 212-216.