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Fixed point theorems with cyclical contractive conditions in b-Menger spaces

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Abstract: In this work, we prove a fixed point theorems in b-Menger spaces (Mbarki et al., Probabilistic b-metric spaces and nonlinear contractions, Fixed Point Theory Appl. 2017, Paper No. 29, 15 p. (2017)) using probabilistic contraction [13] with cyclical conditions. We support our results by an example.

Keywords: b-Menger space, Fixed point, Cyclic probabilistic contraction.

MSC: 47H10,54H25.

1 Introduction

An interesting and important generalization of the notion of metric space was introduced, in 1942, by Menger [11] under the name of statistical metric space, which is now called probabilistic metric space, in this theory, the concept of the distance between two points has a probabilistic nature, i.e., it is exhibited by distribution functions.

Fixed point theory is one of the most famous mathematical theories with application in several branches of sciences, especially in chaos theory, game theory, theory of differential equations etc. The first result from

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the fixed point theory in probabilistic metric spaces was obtained by Sehgal and Bharucha-Reid [9] in 1972 and their fixed point theorem is further generalized by many authors, for example see [3, 5, 8, 9].

The notion of b-metric spaces, as a generalization of metric spaces, was introduced by Bakhtin [1] in 1989 and an extension of Banach's contraction [2] in these spaces was showed by Czerwik [4].

Recently, Mbarki et al., [10] introduced the probabilistic b-metric spaces (b-Menger spaces), as a generalization of probabilistic metric spaces (Menger spaces) and b-metric spaces; and they studied topological structures and properties and showed the fixed point property for nonlinear contractions in these spaces.

On the other hand, cyclic contractions and cyclic contractive type mapping have appeared in several works. This line of research was initiated, in 2003, by Kirk, Srinivasan and Veeramani [7].

In this paper, we prove the existence and uniqueness of the fixed point for the cyclic probabilistic contraction in b-Menger spaces and we give an example which support the main results.

2 Preliminaries

We begin by briefly recalling some definitions and notions from probabilistic b-metric spaces theory that we will use in the sequal. For more details, we refer the reader to [10].

A nonnegative real function f defined on $\mathbb{R}^+ \cup \{\infty\}$ is called a distance distribution function (briefly, a d.d.f) if it is nondecreasing, left continuous on $(0, \infty)$, with $f(0) = 0$ and $f(\infty) = 1$. The set of all d.d.f's will be noted by Δ^+; and the set of all F in Δ^+ for which $\lim_{x \longrightarrow \infty} f(x) = 1$ by D^+.

A simple example of distribution function is Heavyside function in D^+

$$H(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a t-norm if and only if

1. $T(a, 1) = a$, for all $a \in [0, 1]$,
2. $T(0, 0) = 0$.

As examples we mention the three typical examples of continuous t-norms as follows:
$T_p(a, b) = ab$, $T_M(a, b) = Min(a, b)$ and $T_L(a, b) = Max\{a + b - 1, 0\}$.

Definition 2.1. *[10] A b-Menger space is a quadruple (X, F, T, s) where X is a nonempty set, F is a function from $X \times X$ into Δ^+, T is a t-norm, $s \geq 1$ is a real number, and the following conditions are satisfied: for all $x, y, z \in X$ and $\alpha, \beta > 0$,*

1. $F_{xx} = H$,
2. $F_{xy} = H \;\Rightarrow\; x = y$,
3. $F_{xy} = F_{yx}$,
4. $F_{xy}(s(\alpha + \beta)) \geq T(F_{xz}(\alpha), F_{zy}(\beta))$.

It should be noted that a Menger space is a b-Menger with $s = 1$.

The strong topology [12] in a probabilistic semimetric space (i.e., (1), (2) and (3) of Definition 2.1 are satisfied) is introduced by the family of neighborhoods $\wp_x$ of a point $x \in X$ given by

$$\wp_x = \{N_x(\alpha) : \ \alpha > 0\}$$

where

$$N_x(\alpha) = \{y \in X : \ F_{xy}(\alpha) > 1 - \alpha\}.$$

Remark 2.1. *In b-Menger spaces we have the following assertions [10] :*
(a) (M, F, T, s) is endowed with the strong topology is a Hausdorff topological space provided that T is continuous.
(b) The function F is in general not continuous.

In b-Menger space, the convergence of sequence is defined as follows

Definition 2.2. *Let $\{p_n\}$ be a sequence in a b-Menger space (X, F, T, s).*

1. *A sequence $\{p_n\}$ is said to be convergent to $p \in M$, if for every $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $F_{p_n p}(\epsilon) > 1 - \epsilon$ whenever $n \geq N(\epsilon)$.*
2. *A sequence $\{p_n\}$ is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $n, \ m \geq N(\epsilon) \Rightarrow F_{p_n p_m}(\epsilon) > 1 - \epsilon$.*
3. *(X, F, T, s) is said to be complete if every Cauchy sequence has a limit.*

Let f be a self map on X. Power of f at $x \in X$ are defined by $f^0 x = x$ and $f^{n+1} x = f(f^n x)$, $n \geq 0$. We will use the notation $x_n = f^n x$, in particular $x_0 = x$, $x_1 = fx$.
The letter Ψ denotes the set of all function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that

$$0 < \varphi(x) < x \ \ and \ \ lim_{n \longrightarrow \infty} \varphi^n(x) = 0 \ \ for \ \ each \ \ x > 0.$$

Definition 2.3. *[6] We say that a t-norm T is of H-type if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,*

$$\forall \epsilon \in (0, 1), \exists \lambda \in (0, 1) : t > 1 - \lambda \Rightarrow T^n(t) > 1 - \epsilon, \ \ \forall n \geq 1$$

where $T^1(x) = T(x, x)$, $T^n(x) = T(x, T^{n-1}(x))$, for every $n \geq 2$.

The t-norm T_M is a trivial example of t-norm of H-type.

Definition 2.4. *[10] Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi(t) < t$ for $t > 0$, and f be a selfmap of a probabilistic b-metric space (X, F, τ, s). We say that f is φ-probabilistic contraction if*

$$F_{fpfq}(\varphi(t)) \geq F_{pq}(st),$$

for all $p, q \in X$ and $t > 0$,

Lemma 2.1. *[10] Let (X, F, τ_T, s) be a complete probabilistic b-metric space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let $f : X \longrightarrow X$ be a φ-probabilistic contraction where $\varphi \in \Psi$. Then f has a unique fixed point $\overline{u}$, and, for any $u \in X$, $\lim\limits_{n \longrightarrow \infty} f^n(u) = \overline{u}$.*

3 Main results

Befor stating the main fixed point theorems, we need the following concepts.

Definition 3.1. *Let X be a nonempty set, m a positive integer, and $f : X \longrightarrow X$ an operator. $Y = \bigcup_{i=1}^{m} U_i$ is a cyclic representation with respect to f if*

1. *U_i, $i = 1, 2, ..., m$ are nonempty subsets of X,*
2. *$f(U_1) \subset U_2$, $f(U_2) \subset U_3$, ..., $f(U_{m-1}) \subset U_m$, $f(U_m) \subset U_1$.*

Definition 3.2. *Let (X, F, T, s) be a b-Menger space and $f : X \longrightarrow X$ an operator. If*

1. *$Y = \bigcup_{i=1}^{m} U_i$ is a cyclic representation with respect to f,*
2. *$F_{fxfy}(\gamma t) \geq F_{xy}(t)$, for any $x \in U_i$, $y \in U_{i+1}$, and $t > 0$ where $U_{m+1} = U_1$ and $\gamma \in (0, \frac{1}{s})$.*

Then f is called cyclic probabilistic contraction over Y in the b-Menger space (X, F, T, s).

In the proof of our theorems, we use the following lemma:

Lemma 3.1. *Let (X, F, T, s) be a complete b-Menger space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let U and V be two nonempty closed subsets of X and let $f : X \longrightarrow X$ be a cyclic probabilistic contraction over $U \cup V$.*
Then $\{x_n\}$ is a Cauchy sequence for each $x \in U \cup V$.

Proof. Let $x \in U \cup V$, $n \in \mathbb{N}$ and $t > 0$. We claim that

$$F_{x_n x_{n+1}}(t) \geq F_{x_0 x_1}(t\gamma^{-n})$$

For each $n \in \mathbb{N}$, either $x_n \in U$ and $x_{n+1} \in V$ or, $x_n \in V$ and $x_{n+1} \in U$. Then we have

$$\begin{aligned} F_{x_n x_{n+1}}(t) &\geq F_{x_{n-1} x_n}(t\gamma^{-1}) \\ &\geq F_{x_{n-2} x_{n-1}}(t\gamma^{-2}) \\ &\vdots \\ &\geq F_{x_0 x_1}(t\gamma^{-n}). \end{aligned}$$

Next, we show that for each $t > 0$, $n \geq 0$

$$F_{x_n x_m}(t) \geq T^{m-n}(F_{x_0 x_1}((\frac{t}{s} - \gamma t)\gamma^{-n})) \;\; for \;\; all \;\; m > n$$

We put $D_n = F_{x_0 x_1}((\frac{t}{s} - \gamma t)\gamma^{-n})$. Since $\frac{t}{s} - \gamma t > 0$ because $\gamma < \frac{1}{s}$, we have

$$\begin{aligned} F_{x_n x_m}(t) &= F_{x_n x_m}(s(\frac{t}{s} - \gamma t + \gamma t)) \\ &\geq T(F_{x_n x_{n+1}}(\frac{t}{s} - \gamma t), F_{x_{n+1} x_m}(\gamma t)) \\ &\geq T(F_{x_0 x_1}((\frac{t}{s} - \gamma t)\gamma^{-n}), F_{x_{n+1} x_m}(\gamma t)) \\ &= T(D_n, F_{x_{n+1} x_m}(\gamma t)). \end{aligned}$$

By successive application of the above inequality and by associativity and monotonicity of T we have

$$\begin{aligned}
F_{x_n x_m}(t) &\geq T(D_n, T(F_{x_{n+1}x_{n+2}}(\frac{\gamma t}{s} - \gamma^2 t), F_{x_{n+2}x_m}(\gamma^2 t)) \\
&\geq T(D_n, T(F_{x_0 x_1}((\frac{\gamma t}{s} - \gamma^2 t)\gamma^{-(n+1)}), F_{x_{n+2}x_m}(\gamma^2 t)) \\
&= T(D_n, T(F_{x_0 x_1}((\frac{t}{s} - \gamma t)\gamma^{-n}), F_{x_{n+2}x_m}(\gamma^2 t)) \\
&= T(D_n, T(D_n, F_{x_{n+2}x_m}(\gamma^2 t)) \\
&= T(T(D_n, D_n), F_{x_{n+2}x_m}(\gamma^2 t)) \\
&\geq T(T(...T(D_n, D_n), D_n), D_n)..., D_n), F_{x_{m-1}x_m}(\gamma^{m-n-1} t)) \\
&\geq T(T(...T(D_n, D_n), D_n), D_n)..., D_n), F_{x_0 x_1}(\gamma^{-n} t)) \\
&\geq T(T(...T(D_n, D_n), D_n), D_n)..., D_n), D_n) \\
&= T^{m-n}(D_n).
\end{aligned}$$

Since T is a t-norm of H-type, then for arbitrary $\epsilon \in (0,1)$, there exists $\theta = \theta(\epsilon) \in (0,1)$ such that if $t > 1 - \theta$ implies $T^n(t) > 1 - \epsilon$ for all $n \geq 1$.
We have

$$\lim_{n \longrightarrow \infty} (\frac{t}{s} - \gamma t)\gamma^{-n} = \infty.$$

And since $F_{x_0 x_1} \in D^+$, then

$$\lim_{n \longrightarrow \infty} D_n = 1.$$

It follows that there exists $N \in \mathbb{N}$ such that

$$D_n > 1 - \theta \;\; for \;\; all \;\; n > N(\theta(\epsilon)).$$

So

$$F_{x_n x_m}(t) > 1 - \epsilon \;\; for \;\; all \;\; m > n > N.$$

Hence $\{x_n\}$ is a Cauchy sequence in $U \cup V$. □

We now state our main results:

Theorem 3.1. *Let (X, F, T, s) be a complete b-Menger space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let U and V be two nonempty closed subsets of X and let $f : X \longrightarrow X$ be a cyclic probabilistic contraction over $U \cup V$.*
Then f has a unique fixed point in $U \cap V$.

Proof. For $x \in U \cup V$. By Lemma 3.1, $\{x_n\}$ is a Cauchy sequence. Consequently $\{x_n\}$ converges to some point $y \in X$. The subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of $\{x_n\}$ also converge to y. Now, either $\{x_{2n}\} \subset U$ and $\{x_{2n+1}\} \subset V$, or $\{x_{2n}\} \subset V$ and $\{x_{2n+1}\} \subset U$, and since U and V are closed, then $y \in U \cap V$, so $U \cap V \neq \emptyset$. The condition (2) of Definition 3.1 implies that $f : U \cap V \longrightarrow U \cap V$ and the condition (2) of Definition 3.2 implies that f restricted to $U \cap V$ is a probabilistic contraction mapping. By Lemma 2.1 with $\varphi(t) = \gamma s t$, because $\gamma s \in (0,1)$, f has a unique fixed point in $U \cap V$. □

Now, we have the following corollary

Corollary 3.1. *[7] Let U and V be two nonempty closed subsets of a complete metric space (X,d) and let f be a mapping of X into itself satisfying:*
(1) $f(U) \subset V$ and $f(V) \subset U$,
(2) $d(fx,fy) \leq \gamma d(x,y)$, $\forall x \in U$, $\forall y \in V$, where $\gamma \in (0,1)$.
Then f has a unique fixed point in $U \cap V$.

Proof. By [Lemma 3.1, 10], (X,F,T_M) is a complete b-Menger with $s=1$, where

$$F_{xy}(t) = H(t - d(x,y)) \;\; for \;\; all \;\; x,y \in X \;\; and \;\; t > 0.$$

Since $d(fx,fy) \leq \gamma d(x,y)$ we get

$$H(t - d(fx,fy)) \geq H(t - \gamma d(x,y)) \geq H(\frac{1}{\gamma}t - d(x,y)),$$

then

$$F_{fxfy}(\gamma t) \geq F_{xy}(t).$$

By Theorem 3.1 we conclude that f has a unique fixed point in $U \cap V$. □

The reasoning of Theorem 3.1 can be extended to a collection of finite sets

Theorem 3.2. *Let (X,F,T,s) be a complete b-Menger space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let U_i, $i = 1,2,...,m$ be nonempty closed subsets of X and let $f : X \longrightarrow X$ be a cyclic probabilistic contraction over $Y = \bigcup_{i=1}^m U_i$.*
Then f has a unique fixed point in $\bigcap_{i=1}^m U_i$.

Proof. Let $x \in \bigcup_{i=1}^m U_i$. Since $\{x_n\}$ is a Cauchy sequence then $\{x_n\}$ converges to some point y. Each closed subsets U_i, $1 \leq i \leq m$, contain infinitely terms of the sequence $\{x_n\}$, then $y \in U_i$ for all i, $1 \leq i \leq m$, then $y \in \bigcap_{i=1}^m U_i$, so $\bigcap_{i=1}^m U_i \neq \emptyset$. By condition (2) of Definition 3.2, $f : \bigcap_{i=1}^m U_i \longrightarrow \bigcap_{i=1}^m U_i$ is a probabilistic contraction mapping. By Lemma 2.1, f has a unique fixed point in $\bigcap_{i=1}^m U_i$. □

The following example support our results

Example 3.1. *Let $X = [-10,10]$. Define the function $F : X \times X \to \Delta^+$ by*

$$F_{xy}(t) = H(t - |x-y|^2).$$

By [Example 3.3 and Lemma 3.1, 10], $(X,F,T_M,2)$ is a complete b-Menger space, but (X,F,T_M) is not a standard probabilistic metric space because it lacks the triangle inequality:

$$F_{32}(\frac{2}{3}) = 0 < 1 = H(\frac{1}{12}) = Min(F_{3\frac{5}{2}}(\frac{1}{3}), F_{\frac{5}{2}2}(\frac{1}{3})).$$

Let

$$fx = \begin{cases} 9 & if \quad x \in [-10,-4), \\ -\frac{x}{5} & if \quad x \in [-4,4], \\ -9 & if \quad x \in (4,10]. \end{cases}$$

We check that f is not a φ-probabilistic contraction. Assume by contradiction there exists $\varphi : [0,\infty) \to [0,\infty)$ a function such that $\varphi(t) < t$ for $t > 0$, and $F_{fxfy}(\varphi(t)) \geq F_{xy}(2t)$, for all $x, y \in X$ and $t > 0$. It follows that

$$\begin{aligned} 0 = H(83-324) &= F_{f(-\frac{9}{2})f(\frac{9}{2})}(83) \\ &\geq F_{f(-\frac{9}{2})f(\frac{9}{2})}(\varphi(83)) \\ &\geq F_{-\frac{9}{2}\frac{9}{2}}(166)) \\ &= H(166-81) \\ &= 1, \end{aligned}$$

a contradiction. However, if we take $U_1 = [-4,0]$ and $U_2 = [0,4]$. Clearly $f(U_1) \subset U_2$, $f(U_2) \subset U_1$. We claim that

$$H(\frac{1}{25}t - |\frac{x}{5} - \frac{y}{5}|^2) \geq H(t - |x-y|^2) \ \ for \ \ all \ \ x \in U_1 \ \ and \ \ y \in U_2.$$

Indeed, if $t \leq |x-y|^2$ then the inequality is obviously verified, and if $t > |x-y|^2$ then $\frac{1}{25}t > \frac{1}{25}|x-y|^2$, thus $H(t-|x-y|^2) = 1$ and $H(\frac{1}{25}t - |\frac{x}{5} - \frac{y}{5}|^2) = 1$.

So,

$$F_{fxfy}(\frac{1}{25}t) \geq F_{xy}(t) \ \ for \ \ all \ \ x \in U_1 \ \ and \ \ y \in U_2.$$

Hence for $\gamma = \frac{1}{25} \in (0, \frac{1}{2})$, f has a unique fixed point in $U_1 \cap U_2 = \{0\}$, i.e., 0 is the unique fixed point of f.

4 Conclusion

In this paper, we presented the notion of cyclic probabilistic contraction and proved the existence and uniqueness of fixed point for this type mapping in b-Menger space. An example is constructed to support our results.

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