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Fixed point theorems of C-class functions in S_b-metric space

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Abstract: In this paper, we use C-class function in order to obtain some coupled fixed point theorems in the setting of S_b-metric space. Some results are also given in the form of corollaries. Examples are also given to verify our results.

Keywords: Coupled fixed point, S_b metric space and C-class function.
MSC: 47H10, 54H25.

1 Introduction and mathematical preliminaries

S.Sedghi, N. Shobe and A. Aliouche [30] introduced the concept of S-metric space by modifying D-metric and G-metric spaces. Sedhi et. al. [30] remarked that it is easy to see that every D^*-metric is S-metric, but in general the converse is not true. Dhamodharan and krishnakumar [35] introduced the concept of cone $S-$ metric space. Every S metric space is cone S metric space but converse is not true. S. Sedghi and Nguyen van Dung [27] remarked that every S-metric space is topologically equivalent to a b-metric space. Sedghi et.al. [30] claimed that S-metric space is a generalization of a G-metric, that is, every G-metric is an S-metric. But Dung et.al. [32] explained with examples that the assersion of Sedghi et.al [30] is not correct. Dung et.al. [32] further claimed that the class of all S-metric and the class of all G-metric are all distinct. Bakhtin [17] introduced the concept of b-metric space as a generalization of metric space.

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S. Gerwick [19, 18] extended the Banach contraction principle in b-metric space. Inspired by the works of Bakhtin [17] and Sedghi et. al. [30], Nizar and Nabil [33] introduced the concept of S_b-metric space.

Definition 1.1. [33] Let X be a nonempty set and let $b \geq 1$ be a given number. A function $S_b : X^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in X$; the following conditions hold:

(i) $S_b(x, y, z) = 0$ if and only $x = y = z$,

(ii) $S_b(x, x, y) = S_b(y, y, x)$ for all $x, y \in X$,

(iii) $S_b(x, y, z) \leq b[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$

The pair (X, S_b) is called an S_b-metric space.

Remark 1.2. [33] Note that the class of S_b-metric spaces is larger than the class of S-metric spaces. Indeed, every S-metric space is an S_b-metric space with $s = 1$. However, the converse is not always true.

Example 1.3. [33] Let X be a nonempty set and $card(X) \geq 5$. Suppose $X = X_1 \cup X_2$ a partition of X such that $card(X_1) \geq 4$. Let $b \geq 1$. Then

$$S_b(x, y, z) = \begin{cases} 0, & \text{if } x = y = z = 0; \\ 3s, & \text{if } (x, y, z) \in X_1^3; \\ 1, & \text{if } (x, y, z) \notin X_1^3. \end{cases}$$

for all $x, y, z \in X$. S_b is a S_b-metric on X with coefficient $b \geq 1$.

Definition 1.4. [33] Let (X, S_b) be an $S_b$metric space and $\{x_n\}$ be a sequence in X. Then

(i) A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_b(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = z$.

(ii) A sequence $\{x_n\}$ is called Cauchy sequence if and only if $S_b(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) (X, S_b) is said to be a complete S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = \lim_{n\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x)$$

(iv) Define the diameter of a subset Y of X by

$$diam(Y) := Sup\{S_b(x, y, z) | x, y, z \in X\}.$$

Rohen et al. [37] also give the definition of S_b-metric space as follows.

Definition 1.5. Let X be a nonempty set and let $b \geq 1$ be a given number. A function $S : X^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in X$; the following conditions holds:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$,

(ii) $S(x, y, z) \leq b[S(x, x, t) + S(y, y, t) + S(z, z, t)]$

the pair (X, S) is called an S_b-metric space.

Definition 1.6. A S_b-metric S is said to be symmetric if

$S(x, x, y) = S(y, y, x)$ for all $x, y \in X$

In 2014 A.H.Ansari [34] introduced the concept of a C-class functions which cover a large class of contractive conditions.

Definition 1.7. [34]. A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called C-*class* function if for any $s, t \in [0, \infty)$, the following conditions hold:

(i) $F(s, t) \leq s$;

(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $f(0, 0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all C- functions.

Example 1.8. [34]. The following examples show that the class $\mathcal{C}$ is nonempty:

1. $F(s, t) = s - t$.
2. $F(s, t) = ms$, for some $m \in (0, 1)$.
3. $f(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, \infty)$.
4. $f(s, t) = \log(t + a^s)/(1 + t)$, for some $a > 1$.
5. $f(s, t) = \ln(1 + a^s)/2$, for $e > a > 1$. Indeed $f(s, t) = s$ implies that $s = 0$.
6. $f(s, t) = (s + l)^{(1/(1+t)^r)} - l, l > 1$, for $r \in (0, \infty)$.
7. $f(s, t) = s \log_{t+a} a$, for $a > 1$.
8. $f(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
9. $f(s, t) = s\beta(s)$, where $\beta : [0, \infty) \to [0, 1)$ and continuous
10. $f(s, t) = s - \frac{t}{k+t}$.
11. $f(s, t) = s - \varphi(s)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
12. $f(s, t) = sh(s, t)$, where $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$.
13. $f(s, t) = s - (\frac{2+t}{1+t})t$.
14. $f(s, t) = \sqrt[n]{\ln(1 + s^n)}$.
15. $F(s, t) = \phi(s)$, where $\phi : [0, \infty) \to [0, \infty)$ is a upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

16. $F(s,t) = \frac{s}{(1+s)^r}; r \in (0,\infty)$.
17. $F(s,t) = s\pi^{-1/2}\int_0^\infty \frac{e^{-x}}{\sqrt{x+t}}\,dx$.

Let Φ_u denote the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(i) φ is continuous ;

(ii) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.

Definition 1.9. [41] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.10. We let Ψ denote the class of the altering distance functions.

Definition 1.11. [41] A function $\psi : R \to R$ is called total altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.12. We let $\Psi_{\inf}$ denote the class of the total altering distance functions.

Definition 1.13. [41] A tripled (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ is said to be monotone if for any $x, y \in [0,\infty)$

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

Example 1.14. [41] Let $F(s,t) = s - t, \phi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if x} > 1 \end{cases},$$

then (ψ, ϕ, F) is monotone.

Example 1.15. [41] let $F(s,t) = s - t, \phi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if x} > 1 \end{cases},$$

then (ψ, ϕ, F) is not monotone.

2 Main Results

Now we prove the following theorems

Theorem 2.1. *Let (X,S) be a complete symmetric S_b-metric space with parameter $b \geq 1$ and let the mappings $f,g : X^2 \to X$ satisfying*

$$\begin{aligned}
&\psi(S(f(x,y),f(x,y),g(u,v))) \\
&\leq F(\psi(a_1\frac{S(x,x,u)+S(y,y,v)}{2}+a_2\frac{S(f(x,y),f(x,y),g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\
&+a_3\frac{S(f(x,y),f(x,y),g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}+a_4\frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\
&+a_5\frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}+a_6\frac{S(u,u,g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} \\
&+a_7\frac{S(u,u,g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}),\varphi(a_1\frac{S(x,x,u)+S(y,y,v)}{2} \\
&+a_2\frac{S(f(x,y),f(x,y),g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}+a_3\frac{S(f(x,y),f(x,y),g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\
&+a_4\frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}+a_5\frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} \\
&+a_6\frac{S(u,u,g(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}+a_7\frac{S(u,u,g(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)})) \qquad (2.1)
\end{aligned}$$

for all $x,y,u,v \in X$ with $F \in C, \psi \in \Psi, \phi \in \phi_u$ and $a_1,a_2,...,a_7 \geq 0$ with $a_1+a_2+a_3+a_4+a_5+a_6+a_7 = 1$ and $(b < \frac{1-a_2-a_3-a_6-a_7}{a_1+a_4+a_5})$. Then f and g have a unique common coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be arbitrary points.
Define

$$x_{2k+1} = f(x_{2k}, y_{2k}) \quad , \quad y_{2k+1} = f(y_{2k}, x_{2k})$$
$$x_{2k+2} = g(x_{2k+1}, y_{2k+1}) \quad , \quad y_{2k+2} = g(y_{2k+1}, x_{2k+1})$$

for $k = 0,1,2,,....$ Then

$$\begin{aligned}
\psi(S(x_{2k+1},x_{2k+1},x_{2k+2})) &= \psi(S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))) \\
&\leq F(\psi(a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
&+a_2\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
&+a_3\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
&+a_4\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
&+a_5\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
&+a_6\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
&+a_7\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})},
\end{aligned}$$

$$
\begin{aligned}
& \varphi(a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +a_2\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_3\frac{S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_4\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_5\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_6\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_7\frac{S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})})) \\
= \;& F(\psi(a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +a_2\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_3\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_4\frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_5\frac{S(x_{2k},x_{2k},x_{2k+1})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_6\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_7\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}, \\
& \varphi(a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +a_2\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_3\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_4\frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_5\frac{S(x_{2k},x_{2k},x_{2k+1})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_6\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(x_{2k},x_{2k},x_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})} \\
& +a_7\frac{S(x_{2k+1},x_{2k+1},x_{2k+2})S(y_{2k},y_{2k},y_{2k+1})}{1+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})})) \\
\leq \;& F(\psi(a_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2} \\
& +(a_2+a_3)S(x_{2k+1},x_{2k+1},x_{2k+2})+(a_4+a_5)S(x_{2k},x_{2k},x_{2k+1}) \\
& +(a_6+a_7)S(x_{2k+1},x_{2k+1},x_{2k+2}),
\end{aligned}
$$

$$
\begin{aligned}
& \varphi(a_1 \frac{S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}{2} \\
& +(a_2 + a_3)S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + (a_4 + a_5)S(x_{2k}, x_{2k}, x_{2k+1}) \\
& +(a_6 + a_7)S(x_{2k+1}, x_{2k+1}, x_{2k+2}))) \\
\leq \; & a_1 \frac{S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}{2} \\
& +(a_2 + a_3)S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + (a_4 + a_5)S(x_{2k}, x_{2k}, x_{2k+1}) \\
& +(a_6 + a_7)S(x_{2k+1}, x_{2k+1}, x_{2k+2})
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (1 - a_2 - a_3 - a_6 - a_7)S(x_{2k+1}, x_{2k+1}, x_{2k+2}) & \leq (\frac{a_1}{2} + a_4 + a_5)S(x_{2k}, x_{2k}, x_{2k+1}) \\
& + \frac{a_1}{2} S(y_{2k}, y_{2k}, y_{2k+1}) \\
\Rightarrow S(x_{2k+1}, x_{2k+1}, x_{2k+2}) & \leq \frac{\frac{a_1}{2} + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} S(x_{2k}, x_{2k}, x_{2k+1}) \\
& + \frac{\frac{a_1}{2}}{1 - a_2 - a_3 - a_6 - a_7} S(y_{2k}, y_{2k}, y_{2k+1})
\end{aligned} \tag{2.2}
$$

Proceeding similarly one can prove that

$$
\begin{aligned}
\Rightarrow S(y_{2k+1}, y_{2k+1}, y_{2k+2}) & \leq \frac{\frac{a_1}{2} + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} S(y_{2k}, y_{2k}, y_{2k+1}) \\
& + \frac{\frac{a_1}{2}}{1 - a_2 - a_3 - a_6 - a_7} S(x_{2k}, x_{2k}, x_{2k+1})
\end{aligned} \tag{2.3}
$$

Adding (2.2) and (2.3) we have

$$
\begin{aligned}
S(x_{2k+1}, x_{2k+1}, x_{2k+2}) & + S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \\
& \leq \frac{a_1 + a_4 + a_5}{1 - a_2 - a_5 - a_6 - a_7} S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S(x_{2k+1}, x_{2k+1}, x_{2k+2}) & + S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \\
& \leq h[S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})
\end{aligned}
$$

where $h = \frac{a_1 + a_4 + a_5}{1 - a_2 - a_5 - a_6 - a_7}$.

Also, we can show that

$$
\begin{aligned}
S(x_{2k+2}, x_{2k+2}, x_{2k+3}) & + S(y_{2k+2}, y_{2k+2}, y_{2k+3}) \\
& \leq hS(x_{2k+1}, x_{2k+1}, x_{2k+2}) + S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \\
& \leq h^2[S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})]
\end{aligned}
$$

Continuing this way, we have

$$\begin{aligned}
S(x_n, x_n, x_{n+1}) &+ S(y_n, y_n, y_{n+1}) \\
&\leq h[S(x_{n-1}, x_{n-1}, x_n) + S(y_{n-1}, y_{n-1}, y_n)] \\
&\leq h^2[S(x_{n-2}, x_{n-2}, x_{n-1}) + S(y_{n-2}, y_{n-2}, y_{n-1})] \\
&. \\
&. \\
&. \\
&\leq h^n[S(x_0, x_0, x_1) + S(y_0, y_0, y_1)]
\end{aligned}$$

If $S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = S_n$, then for each $n \in \mathbb{N}$. Thus, we conclude that the sequence $\{S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = S_n\}$ is nonnegative and nonincreasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = S_n = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, on account of (2.1), we get that

$$r \leq F(r, \varphi(r)),$$

which yields that $r = 0$ or $\varphi(r) = 0$. We derive

$$r = \lim_{n\to\infty} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = 0. \tag{2.4}$$

$S_n \leq hS_{n-1} \leq h^2 S_{n-2} \leq ... \leq h^n S_0$

So for $m > n$,

$$\begin{aligned}
S(x_n,x_n,x_m)+S(y_n,y_n,y_m) &\leq b[2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m)\\
&\quad +2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m)]\\
&= 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +b[S(x_{n+1},x_{n+1},x_m)+S(y_{n+1},y_{n+1},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +b^2[2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m)\\
&\quad +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m)\\
&= 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +b^2[S(x_{n+2},x_{n+2},x_m)+S(y_{n+2},y_{n+2},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +...+2b^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})]\\
&\quad +b^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +2b^3[S(x_{n+2},x_{n+2},x_{n+3})+S(y_{n+2},y_{n+2},y_{n+3})]\\
&\quad +...+2b^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})]\\
&\quad +2b^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)]\\
&\leq 2\{bh^n+b^2h^{n+1}+b^3h^{n+2}+...+b^{m-n}h^{m-1}\}S_0\\
&< 2bh^n[1+bh+(bh)^2+...]S_0\\
&= \frac{2bh^n}{1-bh}S_0 \to 0 \text{ as } n\to\infty
\end{aligned}$$

which shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. As X is complete S_b-metric space, so there exists $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$.

Now we will prove that $x=f(x,y)$ and $y=f(y,x)$. On the contrary suppose that $x\neq f(x,y)$ and $y\neq f(y,x)$. Then $S(x,x,f(x,y))=l_1>0$ and $S(y,y,f(y,x))=l_2>0$.

Using inequality (2.1)

$$\begin{aligned}
l_1 &= S(x,x,f(x,y)\\
&\leq b[2S(x,x,x_{n+1})+S(x_{n+1},x_{n+1},f(x,y))]\\
&= b[2S(x,x,x_{n+1})+S(f(x_n,y_n),f(x_n,y_n),f(x,y))]\\
&\leq 2bS(x,x,x_{n+1})+F(\psi(b\Big[a_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2}\\
&\quad +a_2\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&\quad +a_3\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}
\end{aligned}$$

$$\begin{aligned}
&+a_4\frac{S(x_n,x_n,f(x_n,y_n))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_5\frac{S(x_n,x_n,f(x_n,y_n))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_6\frac{S(x,x,f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_7\frac{S(x,x,f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\Big],\\
&\varphi(b\Big[a_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2}\\
&+a_2\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_3\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_4\frac{S(x_n,x_n,f(x_n,y_n))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_5\frac{S(x_n,x_n,f(x_n,y_n))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_6\frac{S(x,x,f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_7\frac{S(x,x,f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\Big]))\\
\leq\ & 2bS(x,x,x_{n+1})+b\Big[a_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2}\\
&+a_2\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_3\frac{S(f(x_n,y_n),f(x_n,y_n),f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_4\frac{S(x_n,x_n,f(x_n,y_n))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}+a_5\frac{S(x_n,x_n,f(x_n,y_n))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\\
&+a_6\frac{S(x,x,f(x,y))S(x_n,x_n,x)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}+a_7\frac{S(x,x,f(x,y))S(y_n,y_n,y)}{1+S(x_n,x_n,x)+S(y_n,y_n,y)}\Big]
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, therefore by taking limit as $n\to\infty$ we get $l_1\leq 0$, which is a contradiction, so $S(x,x,f(x,y))=0$ which gives $x=f(x,y)$.

Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (2.1), we have

$$\begin{aligned}
\psi(S(x,x,p)) &= \psi(S(f(x,y),f(x,y),g(p,q))) \\
&\leq F(\psi(\frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2\frac{S(f(x,y),f(x,y),g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&\quad +a_3\frac{S(f(x,y),f(x,y),g(p,q))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_4\frac{S(x,x,f(x,y))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}
\end{aligned}$$

$$\begin{aligned}
&+a_5\frac{S(x,x,f(x,y))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_6\frac{S(p,p,g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_7\frac{S(p,p,g(p,q))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)},\varphi(\frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
&+a_2\frac{S(f(x,y),f(x,y),g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_3\frac{S(f(x,y),f(x,y),g(p,q))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_4\frac{S(x,x,f(x,y))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_5\frac{S(x,x,f(x,y))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_6\frac{S(p,p,g(p,q))S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_7\frac{S(p,p,g(p,q))S(y,y,q)}{1+S(x,x,p)+S(y,y,q)})) \\
= \ &F(\psi(\frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2\frac{S(x,x,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_3\frac{S(x,x,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_4\frac{S(x,x,x)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_5\frac{S(x,x,x)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_6\frac{S(p,p,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_7\frac{S(p,p,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)},\varphi(\frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
&+a_2\frac{S(x,x,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_3\frac{S(x,x,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_4\frac{S(x,x,x)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_5\frac{S(x,x,x)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_6\frac{S(p,p,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)}+a_7\frac{S(p,p,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)})) \\
&\leq \psi(\frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2\frac{S(x,x,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_3\frac{S(x,x,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_4\frac{S(x,x,x)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_5\frac{S(x,x,x)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)}+a_6\frac{S(p,p,p)S(x,x,p)}{1+S(x,x,p)+S(y,y,q)} \\
&+a_7\frac{S(p,p,p)S(y,y,q)}{1+S(x,x,p)+S(y,y,q)})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow S(x,x,p) &\leq \frac{a_1}{2}\{S(x,x,p)+S(y,y,q)\}+a_2S(x,x,p)+a_3S(x,x,p) \\
\Rightarrow (1-a_2-a_3)S(x,x,p) &\leq \frac{a_1}{2}S(x,x,p)+\frac{a_1}{2}S(y,y,q)
\end{aligned}$$

$$\Rightarrow (1 - \frac{a_1}{2} - a_2 - a_3)S(x,x,p) \leq \frac{a_1}{2}S(y,y,q)$$
$$\Rightarrow S(x,x,p) \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3}S(y,y,q) \tag{2.5}$$

Similarly,

$$S(y,y,q) \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3}S(x,x,p) \tag{2.6}$$

Adding (2.5) and (2.6) we have

$$S(x,x,p) + S(y,y,q) \leq \frac{a_1}{2 - a_1 - 2a_2 - 2a_3}[S(x,x,p) + S(y,y,q)]$$
$$\Rightarrow [1 - \frac{a_1}{2 - a_1 - 2a_2 - 2a_3}][S(x,x,p) + S(y,y,q)] \leq 0$$
$$\Rightarrow \frac{2(1 - a_1 - a_2 - a_3)}{2 - a_1 - 2a_2 - a_3}[S(x,x,p) + S(y,y,q)] \leq 0$$

Since $a_1 + a_2 + a_3 < 1$, $\frac{2(1-a_1-a_2-a_3)}{2-a_1-2a_2-2a_3} > 0$.

Hence $S(x,x,p) + S(y,y,q) = 0$,

which implies that $x = p$ and $y = q \Rightarrow (x,y) = (p,q)$.

Thus f and g have unique coupled common fixed point. This completes the proof. □

Corollary 2.2. *Let (X,S) be a complete symmetric S_b-metric space with parameter $b \geq 1$ and let the mapping $f : X^2 \to X$ satisfying*

$$\psi(S(f(x,y),f(x,y),f(u,v)))$$
$$\leq F(\psi(a_1\frac{S(x,x,u)+S(y,y,v)}{2} + a_2\frac{S(f(x,y),f(x,y),f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}$$
$$+a_3\frac{S(f(x,y),f(x,y),f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} + a_4\frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}$$
$$+a_5\frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)} + a_6\frac{S(u,u,f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)}$$
$$+a_7\frac{S(u,u,f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}, \varphi(a_1\frac{S(x,x,u)+S(y,y,v)}{2}$$
$$+a_2\frac{S(f(x,y),f(x,y),f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} + a_3\frac{S(f(x,y),f(x,y),f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}$$
$$+a_4\frac{S(x,x,f(x,y))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} + a_5\frac{S(x,x,f(x,y))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}$$
$$+a_6\frac{S(u,u,f(u,v))S(x,x,u)}{1+S(x,x,u)+S(y,y,v)} + a_7\frac{S(u,u,f(u,v))S(y,y,v)}{1+S(x,x,u)+S(y,y,v)}))$$

for all $x,y,u,v \in X$ with $F \in C, \psi \in \Psi, \phi \in \phi_u$ and $a_1, a_2, ..., a_7 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 < 1$. Then f has a unique coupled fixed point in X.

Theorem 2.3. *Let (X,S) be a complete symmetric S_b-metric space with parameter $b \geq 1$ and let the mappings*

$f, g : X^2 \to X$ satisfy

$$\begin{aligned}\psi(S(f(x,y),f(x,y),g(u,v))) \leq\ & F(\psi(\beta_1\frac{S(x,x,u)+S(y,y,v)}{2}\\ &+\beta_2\frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+s[S(x,x,g(x,y)+S(u,u,f(u,v))+S(x,x,u)+S(y,y,v))]}\\ &\varphi(\beta_1\frac{S(x,x,u)+S(y,y,v)}{2}\\ &+\beta_2\frac{S(x,x,f(x,y))S(u,u,g(u,v))}{1+s[S(x,x,g(x,y)+S(u,u,f(u,v))+S(x,x,u)+S(y,y,v))]}))\end{aligned} \tag{2.7}$$

for all $x, y, u, v \in X$ with $F \in C, \psi \in \Psi, \phi \in \phi_u$ and β_1, β_2 are non-negative real numbers with $\beta_1 + \beta_2 < 1$ and $b < \frac{1-\beta_2}{\beta_1}$. Then f and g have unique common coupled fixed point.

Proof. Let x_0, y_0 be arbitrary points. Define

$$x_{2k+1} = f(x_{2k}, x_{2k}) \quad , \quad y_{2k+1} = f(y_{2k}, x_{2k})$$
$$x_{2k+2} = g(x_{2k+1}, y_{2k+1}) \quad , \quad y_{2k+2} = g(y_{2k+1}, x_{2k+1})$$

for $k = 0, 1, 2, ...$ Then

$$\begin{aligned}&\psi(S(x_{2k+1},x_{2k+1},x_{2k+2}))\\ &=\psi(S(f(x_{2k},y_{2k}),f(x_{2k},y_{2k}),g(x_{2k+1},y_{2k+1})))\\ &\leq F(\psi(\beta_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}\\ &+\beta_2\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))}{1+s[S(x_{2k},x_{2k},g(x_{2k+1},y_{2k+1}))+S(x_{2k+1},x_{2k+1},f(x_{2k},y_{2k}))+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]},\\ &\varphi(\beta_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}\\ &+\beta_2\frac{S(x_{2k},x_{2k},f(x_{2k},y_{2k}))S(x_{2k+1},x_{2k+1},g(x_{2k+1},y_{2k+1}))}{1+s[S(x_{2k},x_{2k},g(x_{2k+1},y_{2k+1}))+S(x_{2k+1},x_{2k+1},f(x_{2k},y_{2k}))+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]}))\\ &=F(\psi(\beta_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}\\ &+\beta_2\frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k+1},x_{2k+1},x_{2k+2})}{1+s[S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k+1},x_{2k+1},x_{2k+1})+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]},\\ &\varphi(\beta_1\frac{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})}{2}\\ &+\beta_2\frac{S(x_{2k},x_{2k},x_{2k+1})S(x_{2k+1},x_{2k+1},x_{2k+2})}{1+s[S(x_{2k},x_{2k},x_{2k+2})+S(x_{2k+1},x_{2k+1},x_{2k+1})+S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]}))\\ &\leq F(\psi(\frac{\beta_1}{2}\{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})\}+\beta_2 S(x_{2k+1},x_{2k+1},x_{2k+2}),\\ &\varphi(\frac{\beta_1}{2}\{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})\}+\beta_2 S(x_{2k+1},x_{2k+1},x_{2k+2})))\\ &\leq\frac{\beta_1}{2}\{S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})\}+\beta_2 S(x_{2k+1},x_{2k+1},x_{2k+2})\end{aligned}$$

$$\begin{aligned}\Rightarrow (1-\beta_2)S(x_{2k+1},x_{2k+1},x_{2k+2}) &\leq \frac{\beta_1}{2}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]\\ \Rightarrow S(x_{2k+1},x_{2k+1},x_{2k+2}) &\leq \frac{\beta_1}{2(1-\beta_2)}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]\end{aligned} \tag{2.8}$$

Similarly we can show that

$$S(y_{2k+1},y_{2k+1},y_{2k+2}) \leq \frac{\beta_1}{2(1-\beta_2)}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})] \tag{2.9}$$

Adding (2.8) and (2.9) we have

$$\begin{aligned}S(x_{2k+1},x_{2k+1},x_{2k+2}) &+ S(y_{2k+1},y_{2k+1},y_{2k+2})\\ &\leq \frac{\beta_1}{1-\beta_2}[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]\\ &= k[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]\end{aligned}$$

where $k=\frac{\beta_1}{1-\beta_2}$.
Similarly, we can show that

$$\begin{aligned}S(x_{2k+2},x_{2k+2},x_{2k+3}) &+ S(y_{2k+2},y_{2k+2},y_{2k+3})\\ &\leq k[S(x_{2k+1},x_{2k+1},x_{2k+2})+S(y_{2k+1},y_{2k+1},y_{2k+2})]\\ &\leq k^2[S(x_{2k},x_{2k},x_{2k+1})+S(y_{2k},y_{2k},y_{2k+1})]\end{aligned}$$

If $S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1}) = S_n$, then for each $n \in \mathbb{N}$, we conclude that the sequence $\{S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1}) = S_n\}$ is nonnegative and nonincreasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1}) = S_n = r$. We claim that $r=0$. Suppose, on the contrary, that $r>0$. Then, on account of (2.7), we get that

$$r \leq F(r,\varphi(r)),$$

which yields that $r=0$, or , $\varphi(r)=0$. We derive

$$r = \lim_{n\to\infty} S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1}) = 0. \tag{2.10}$$

$$S_n \leq kS_{n-1} \leq k^2S_{n-2} \leq ... \leq k^nS_0$$

So, for $m > n$ we have

$$\begin{aligned}
S(x_n,x_n,x_m)+S(y_n,y_n,y_m) &\leq b[2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m)\\
&\quad +2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m)]\\
&= b[2S(x_n,x_n,x_{n+1})+2S(y_n,y_n,y_{n+1})]\\
&\quad +b[S(x_{n+1},x_{n+1},x_m)+S(y_{n+1},y_{n+1},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +b^2[2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m)\\
&\quad +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m)\\
&= 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +b[S(x_{n+2},x_{n+2},x_m)+S(y_{n+2},y_{n+2},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +...+2b^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})]\\
&\quad +b^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)]\\
&\leq 2b[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]\\
&\quad +2b^2[S(x_{n+1},x_{n+1},x_{n+2})+S(y_{n+1},y_{n+1},y_{n+2})\\
&\quad +...+2b^{m-n-1}[S(x_{m-2},x_{m-2},x_{m-1})+S(y_{m-2},y_{m-2},y_{m-1})]\\
&\quad +2b^{m-n}[S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)]\\
&\leq 2\{bk^n+b^2k^{n+1}+...+b^{m-n}k^{m-1}\}S_0\\
&< 2bk^n[1+bk+(bk)^2+...]S_0\\
&= \frac{2bk^n}{1-bk}S_0 \to 0 \text{ as } n\to\infty
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete S_b-metric space, there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now, we will show that $x = f(x,y)$ and $y = f(y,x)$. Suppose on contrary that $x \neq f(x,y)$ and $y \neq f(y,x)$, so that $S(x,x,f(x,y)) = l_1 > 0$ and $S(y,y,f(y,x)) = l_2 > 0$. Consider the following and using inequality (2.7), we get

$$\begin{aligned}
\psi(l_1) &= \psi(S(x,x,f(x,y)) \\
&\leq \psi(b[2S(x,x,x_{n+1})+S(x_{n+1},x_{n+1},f(x,y))]) \\
&\leq \psi(bS(x,x,x_{n+1}))+\psi(bS(f(x_n,y_n),f(x_n,y_n),f(x,y))) \\
&\leq \psi(bS(x,x,x_{n+1}))+F(\psi(b[\beta_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2} \\
&\quad +\beta_2\frac{S(x_n,x_n,f(x_n,y_n))S(x,x,f(x,y))}{1+b[S(x_n,x_n,f(x,y))+S(x,x,f(x_n,y_n))+S(x_n,x_n,x)+S(y_n,y_n,y)]}], \\
&\quad \varphi(s[\beta_1\frac{S(x_n,x_n,x)+S(y_n,y_n,y)}{2} \\
&\quad +\beta_2\frac{S(x_n,x_n,f(x_n,y_n))S(x,x,f(x,y))}{1+b[S(x_n,x_n,f(x,y))+S(x,x,f(x_n,y_n))+S(x_n,x_n,x)+S(y_n,y_n,y)]}])) \\
&= \psi(bS(x,x,x_{n+1}))+F(\psi(\frac{b\beta_1}{2}[S(x_n,x_n,x)+S(y_n,y_n,y)] \\
&\quad +\beta_2\frac{S(x_n,x_n,x_{n+1})S(x,x,f(x,y))}{1+b[S(x_n,x_n,f(x,y))+S(x,x,x_{n+1})+S(x_n,x_n,x)+S(y_n,y_n,y)]}, \\
&\quad \frac{b\beta_1}{2}[S(x_n,x_n,x)+S(y_n,y_n,y)] \\
&\quad +\beta_2\frac{S(x_n,x_n,x_{n+1})S(x,x,f(x,y))}{1+b[S(x_n,x_n,f(x,y))+S(x,x,x_{n+1})+S(x_n,x_n,x)+S(y_n,y_n,y)]})
\end{aligned}$$

Taking the limit as $n \to \infty$ we get

$$S(x,x,f(x,y)) \leq 0$$

Therefore

$$S(x,x,f(x,y)) = 0$$

which implies that $x = f(x,y)$. Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence, (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the common coupled fixed point of f and g, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (2.7), we have

$$\begin{aligned}
S(x,x,p) &= S(f(x,y),f(x,y),g(p,q)) \\
&\leq \frac{\beta_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
&\quad +\beta_2\frac{S(x,x,f(x,y))S(p,p,g(p,q))}{1+s[S(x,x,g(p,q))+S(p,p,f(x,y))+S(x,x,p)+S(y,y,q)]}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow S(x,x,p) &\leq \frac{\beta_1}{2}\{S(x,x,p)+S(y,y,q)\} \\
\Rightarrow (1-\frac{\beta_1}{2})S(x,x,p) &\leq \frac{\beta_1}{2}S(y,y,q) \\
\Rightarrow S(x,x,p) &\leq \frac{\beta_1}{2-\beta_1}S(y,y,q) \qquad (2.11)
\end{aligned}$$

Similarly

$$S(y,y,q) \leq \frac{\beta_1}{2-\beta_1}S(x,x,p) \qquad (2.12)$$

Adding (2.11) and (2.12) we have

$$\begin{aligned} S(x,x,p)+S(y,y,q) &\leq \frac{\beta_1}{2-\beta_1}[S(x,x,p)+S(y,y,q)] \\ \Rightarrow (1-\frac{\beta_1}{2-\beta_1})[S(x,x,p)+S(y,y,q)] &\leq 0 \\ \Rightarrow \frac{2(1-\beta_1)}{2-\beta_1}[S(x,x,p)+S(y,y,q)] &\leq 0 \end{aligned}$$

But $\frac{2(1-\beta_1)}{2-\beta_1} > 0$. Therefore $S(x,x,p)+S(y,y,q) = 0$. Which implies that $x = p$ and $y = q \Rightarrow (x,y) = (p,q)$. Thus f and g have a unique common coupled fixed point. □

This completes the proof.

Corollary 2.4. *Let (X,S) be a complete symmetric S_b-metric space with parameter $b \geq 1$ and let the mapping $f : X^2 \to X$ satisfying*

$$\begin{aligned} \psi(S(f(x,y),f(x,y),f(u,v))) \leq\ & F(\psi(\beta_1\frac{S(x,x,u)+S(y,y,v)}{2} \\ &+\beta_2\frac{S(x,x,f(x,y))S(u,u,f(u,v))}{1+s[S(x,x,f(u,v)+S(u,u,f(x,y))+S(x,x,u)+S(y,y,v))]}, \\ &\varphi(\beta_1\frac{S(x,x,u)+S(y,y,v)}{2} \\ &+\beta_2\frac{S(x,x,f(x,y))S(u,u,f(u,v))}{1+s[S(x,x,f(u,v)+S(u,u,f(x,y))+S(x,x,u)+S(y,y,v))]})) \end{aligned}$$

for all $x,y,u,v \in X$ with $F \in C$ $\psi \in \Psi$, $\phi \in \Phi_u$ and β_1,β_2 are non-negative real numbers with $\beta_1+\beta_2 < 1$. Then f has a unique coupled fixed point.

Example 2.5. let $\mathbb{R}$ be the real line. Then

$$S(x,y,z) = |x-z| + |y-z| \text{ for all } x,y,z \in \mathbb{R}$$

is an S_b-metric on $\mathbb{R}$.

Let $f(x,y) = g(x,y) = \frac{2x-y+11}{12}$, $b = 2$, $\psi(t) = t$ and $\varphi(t) = t$

Then the pair (f,g) has the mixed weakly monotone property and

$$\begin{aligned} \psi(S(f(x,y),f(x,y),g(u,v))) &= F(\psi(2\left|\frac{2x-y+11}{12}-\frac{2u-v+11}{12}\right|),\varphi(2\left|\frac{2x-y+11}{12}-\frac{2u-v+11}{12}\right|)) \\ &= F(2\left|\frac{2x-y+11}{12}-\frac{2u-v+11}{12}\right|,2\left|\frac{2x-y+11}{12}-\frac{2u-v+11}{12}\right|) \\ &= F(\frac{|x-u|}{3}+\frac{|y-v|}{6},\frac{|x-u|}{3}+\frac{|y-v|}{6}) \\ &\leq F(\frac{1}{3}(|x-u|+|y-v|),\frac{1}{3}(|x-u|+|y-v|)) \\ &= F(\frac{1}{3}\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right),\frac{1}{3}\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right)) \\ &\leq \frac{1}{3}\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right) \end{aligned}$$

In equation (2.1) is satisfied with $a_1 = \frac{1}{3}$ and $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$. Note that corollary (2.2) is also satisfied.

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