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Coupled fixed point theorems for generalized contractions in ordered M-metric spaces

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Abstract: Using control functions, we improve and extend some results of coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces by Lakshmikantam and Ćirić [5] to ordered M-metric spaces.

Keywords: Fixed point; Partial metric space; M-metric space; Coupled fixed point.
MSC: 47H10, 54H25.

1 Introduction and Preliminaries

In 2014 Asadi *et al.* [1] introduced the M-metric space which extends p-metric space. Many authors proved fixed point theorems on M-metric spaces, see [1, 2, 3, 4, 7]. Lakshmikantam and Ćirić in [5], gave some interesting fixed point results on ordered metric spaces. Now, by using a class of control functions, we extend main results of [5] to complete M-metric spaces. We mention two important remark of [5] as well.

About the definition, topology of partial metric, convergence, Cauchy sequence on (X, p) and more detail, refer to [6].

A new extension of the Definition of the partial metric is came as follows:

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Definition 1.1. ([1]) *Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a m-metric if the following conditions are satisfied:*

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,

(m2) $m_{xy} \leq m(x,y)$, *where* $m_{xy} := \min\{m(x,x), m(y,y)\}$,

(m3) $m(x,y) = m(y,x)$,

(m4) $\left(m(x,y) - m_{xy}\right) \leq \left(m(x,z) - m_{xz}\right) + \left(m(z,y) - m_{zy}\right)$.

Then the pair (X,m) is called an M-metric space.

We note that every p-metric is a m-metric. In [1] you can find an example of a m-metric which is not p-metric.

For more examples and results which obtained in M-metric spaces refer to [1, 2, 3, 7].
Some of fundamental notations are useful in the sequel.
For convergent:

$$x_n \to x \iff m(x_n, x) - m_{x_n,x} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.1}$$

For m-Cauchy sequence:

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_n,x_m}) < \infty \quad \text{and} \quad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) < \infty, \tag{1.2}$$

where $M_{xy} := \max\{m(x,x), m(y,y)\}$.

Theorem 1.2. *Suppose that $f : X \to Y$ is a map between M-metric spaces (X,m) and (Y,m'). Then f is continuous if and only if it is sequentially continuous.*

Proof. Suppose that f is continuous in x_0 and $\epsilon > 0$. Then

$$\exists \delta > 0 \quad f(B_m(x_0,\delta)) \subseteq B_{m'}(f(x_0),\epsilon).$$

Therefore,

$$m(x,x_0) < m_{x,x_0} + \delta \Longrightarrow m'(fx, fx_0) < m'_{fx,fx_0} + \epsilon.$$

Suppose $x_n \to x_0$ in τ_m topology. So, $\lim_{n\to\infty}(m(x_n,x_0) - m_{x_n,x_0}) = 0$, then

$$\exists N \in \mathbb{N}, \forall n \geq N, \mid m(x_n,x_0) - m_{x_n,x_0} \mid < \delta.$$

Therefore, $m(x_n,x_0) < m_{x_n,x_0} + \delta$. So,

$$m'(fx_n, fx_0) - m'_{fx_n,fx_0} < \epsilon.$$

On the other hand, $m'_{fx_n,fx_0} \leq m'(fx_n, fx_0)$ and then

$$\mid m'(fx_n, fx_0) - m'_{fx_n,fx_0} \mid < \epsilon,$$

so

$$\lim_{n\to\infty}(m'(fx_n, fx_0) - m'_{fx_n,fx_0}) = 0$$

therefore, $fx_n \to fx_0$ in τ_m topology.

Now, in contrary, suppose f is sequentially continuous at x_0 and is not continuous in x_0. Then

$$\exists\, \epsilon_0 > 0, \forall\, n \geq 1, \exists x_n \in X, m(x_n, x_0) < m_{x_n,x_0} + \frac{1}{n}, \tag{1.3}$$

$$m'(fx_n, fx_0) \geq m'_{fx_n,fx_0} + \epsilon_0.$$

So,

$$\mid m(x_n, x_0) - m_{x_n,x_0} \mid = m(x_n, x_0) - m_{x_n,x_0} < \frac{1}{n},$$

therefore, $(m(x_n, x_0) - m_{x_n,x_0}) \to 0$ then $x_n \to x_0$ as $n \to \infty$ in τ_m topology, that implies $fx_n \to fx_0$ in τ_m topology. So ,

$$(m'(fx_n, fx_0) - m'_{fx_n,fx_0}) \to 0 \text{ as } n \to \infty,$$

contradicts with (1.3). Therefore, f is continuous at x_0. □

Remark 1.3. *Similar to proof of the above theorem, we can prove that a function $f : X \times X \to Y$ is continuous at (x_0, y_0) if it is sequentially continuous at (x_0, y_0).*

Definition 1.4. *Let $(X, \leq)$ be a partially ordered set and $f : X \times X \to X$ and $g : X \to X$. We say*

1. *f has the mixed g-monotone property if*

$$g(x_1) \leq g(x_2) \ \text{ implies } \ f(x_1, y) \leq f(x_2, y) \tag{1.4}$$

and

$$g(y_1) \leq g(y_2) \ \text{ implies } \ f(x, y_1) \leq f(x, y_2) \tag{1.5}$$

for every $x, y, x_1, x_2, y_1, y_2 \in X$.

2. *An element (x, y) is called a coupled coincidence point of mappings f and g if*

$$f(x, y) = g(x) \quad \text{and} \quad f(y, x) = g(y).$$

3. *f and g are commutative if*

$$g(f(x, y)) = f(g(x), g(y)) \quad \forall x, y \in X.$$

For simplicity, we denote $g(x)$ by gx.

Theorem 1.5. [5] Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and

$$d(F(x, y), F(u, v)) \leq \varphi(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2})$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n.

(ii) if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x),$$

that is, F and g have a coupled coincidence.

2 main results

Definition 2.1. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\psi(t) \to 0$ if and only if $t \to 0$, ψ^{-1} is nondecreasing and one to one and $\psi(a+b) \leq \psi(a) + \psi(b)$ (sub-additivity) for $a, b \in \mathbb{R}^+$. We denote the set of these functions by Ψ. For example, $\psi(t) = \alpha t$ and $\psi(t) = e^{\alpha t}$ for $\alpha \geq 0$ belong to Ψ.*

Theorem 2.2. *Let $(X, m, \leq)$ be a partially ordered complete M-metric space. Assume that there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$, $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $f : X \times X \to X$ and $g : X \to X$ are such that f has the mixed g-monotone property and*

$$\psi(m(f(x,y), f(u,v))) \leq \varphi(\frac{\psi(m(gx, gu)) + \psi(m(gy, gv))}{2}) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $gx \leq gu$ and $gy \geq gv$ and $\psi \in \Psi$. Suppose $f(X \times X) \subseteq g(X)$, g is continuous and commute with f and also suppose either
(a) f is continuous or
(b) X has the following property:

(i) if for a non-decreasing sequence $m^w(x_n, x) \to 0$, then

$$gx_n \leq gx \quad \forall n. \tag{4}$$

(ii) if for a non-increasing sequence $m^w(y_n, y) \to 0$, then

$$gy \leq gy_n \quad \forall n. \tag{5}$$

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq f(x_0, y_0) \quad \text{and} \quad g(y_0) \geq f(y_0, x_0),$$

then f and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \leq f(x_0, y_0)$ and $gy_0 \geq f(y_0, x_0)$. By $f(X \times X) \subseteq g(X)$, there exist $x_1, y_1 \in X$ such that $f(x_0, y_0) = g(x_1)$ and $f(y_0, x_0) = g(y_1)$. Again from $f(X \times X) \subseteq g(X)$ we can choose

$x_2, y_2 \in X$ such that $f(x_1, y_1) = g(x_2)$ and $f(y_1, x_1) = g(y_2)$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$f(x_n, y_n) = g(x_{n+1}) \quad \text{and} \quad f(y_n, x_n) = g(y_{n+1}) \quad \forall n \geq 0. \tag{2.2}$$

By mathematical induction, we show

$$g(x_n) \leq g(x_{n+1}), \quad n \geq 0 \tag{2.3}$$

and

$$g(y_n) \geq g(y_{n+1}), \quad n \geq 0. \tag{2.4}$$

Let $n = 0$. Since $g(x_0) \leq f(x_0, y_0)$ and $g(y_0) \geq f(y_0, x_0)$, and as $g(x_1) = f(x_0, y_0)$ and $g(y_1) = f(y_0, x_0)$, we have $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$. Thus (7) and (8) hold for $n = 0$. Now, suppose that (2.3) and (2.4) hold for some $n \geq 0$. Then, since $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$, and as f has the mixed g-monotone property, from (2.2) and (1.4)

$$g(x_{n+1}) = f(x_n, y_n) \leq f(x_{n+1}, y_n) \text{ and } f(y_{n+1}, x_n) \leq f(y_n, x_n) = g(y_{n+1}) \tag{2.5}$$

and from (2.2) and (1.5)

$$g(x_{n+2}) = f(x_{n+1}, y_{n+1}) \geq f(x_{n+1}, y_n) \text{ and } f(y_{n+1}, x_n) \geq f(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{10}$$

Now from (2.5) and ((10)), we get

$$g(x_{n+1}) \leq g(x_{n+2})$$

and

$$g(y_{n+1}) \geq g(y_{n+2}).$$

Thus (2.3) and (2.4) hold for all $n \geq 0$. Denote

$$\alpha_n = \psi(p(gx_n, gx_{n+1})) + \psi(p(gy_n, gy_{n+1})),$$

for any $n \geq 0$. We show that

$$\alpha_n \leq 2\varphi(\frac{\alpha_{n-1}}{2}). \tag{2.6}$$

Since $gx_{n-1} \leq gx_n$ and $gy_{n-1} \geq gy_n$, from (2.2) and (2.1)

$$\begin{aligned} \psi(m(gx_n, gx_{n+1})) &= \psi(m(f(x_{n-1}, y_{n-1}), f(x_n, y_n))) \\ &\leq \varphi(\frac{\psi(m(gx_{n-1}, gx_n)) + \psi(m(gy_{n-1}, gy_n))}{2}) \\ &= \varphi(\frac{\alpha_{n-1}}{2}). \end{aligned} \tag{2.7}$$

Similarly, from (2.2) and (2.1), as $gy_n \leq gy_{n-1}$ and $gx_n \geq gx_{n-1}$

$$\begin{aligned} \psi(m(gy_{n+1}, gy_n)) &= \psi(m(f(y_n, x_n), f(y_{n-1}, x_{n-1}))) \\ &\leq \varphi(\frac{\psi(m(gy_{n-1}, gy_n)) + \psi(m(gx_{n-1}, gx_n))}{2}) \\ &= \varphi(\frac{\alpha_{n-1}}{2}). \end{aligned}$$

(2.8)

We add (2.7) and (2.8) and obtain (2.6). Since $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$, we have $\varphi(t) \leq t$ for $t \geq 0$. So, (2.6) implies that $\{\alpha_n\}$ is monotone decreasing. Therefore, there is some $\alpha \geq 0$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. We show that $\alpha = 0$. Suppose to the contrary that $\alpha > 0$. Then, taking the limit of both sides of (2.6) as $n \to \infty$ and keep in mind that we assume $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$, we have

$$\alpha = \lim_{n\to\infty} \alpha_n \leq 2 \lim_{n\to\infty} \varphi(\frac{\alpha_{n-1}}{2}) = 2 \lim_{\alpha_{n-1}\to\alpha} \varphi(\frac{\alpha_{n-1}}{2}) < 2\frac{\alpha}{2} = \alpha,$$

a contradiction. Thus $\alpha = 0$, that is

$$\lim_{n\to\infty} [\psi(m(gx_n, gx_{n+1})) + \psi(m(gy_n, gy_{n+1}))] = 0. \tag{2.9}$$

Then

$$m(gx_n, gx_{n+1}) \to 0 \quad \text{and} \quad m(gy_n, gy_{n+1}) \to 0. \tag{2.10}$$

Also we have

$$0 \leq m_{gx_n,gx_{n+1}} \leq m(gx_n, gx_{n+1}) \Rightarrow \lim_{n\to\infty} m_{gx_n,gx_{n+1}} = 0,$$

and

$$m_{gx_n,gx_{n+1}} = \min\{m(gx_n, gx_n), m(gx_{n+1}, gx_{n+1})\} \Rightarrow \lim_{n\to\infty} m(gx_n, gx_n) = 0.$$

On the other hand

$$m_{gx_n,gx_m} = \min\{m(gx_n, gx_n), m(gx_m, gx_m)\} \Rightarrow \lim_{n,m\to\infty} m_{gx_n,gx_m} = 0,$$

so

$$\lim_{n,m\to\infty} (M_{gx_n,gx_m} - m_{gx_n,gx_m}) = 0.$$

We show

$$\lim_{n,m\to\infty} (m(gx_n, gx_m) - m_{gx_n,gx_m}) = 0.$$

Define

$$M(x, y) := m(x, y) - m_{x,y}, \quad \forall x, y \in X.$$

By $(m2)$

$$m_{gx_n,gx_n} \to 0 \quad \text{and} \quad m_{gy_n,gy_n} \to 0. \tag{2.11}$$

Now we show that $\lim_{m,n\to\infty} m(gx_n, gx_m) = \lim_{m,n\to\infty} m(gy_n, gy_m) = 0$. Suppose to the contrary, they are not true. Then there exist an $\varepsilon_0 > 0$ and two subsequences of integers $\{\ell(k)\}$ and $\{m(k)\}$ with $m(k) > \ell(k) \geq k$ and

$$r_k = \psi(M(g(x_{\ell(k)}), g(x_{m(k)}))) + \psi(M(g(y_{\ell(k)}), g(y_{m(k)}))) \geq \varepsilon_0 \tag{2.12}$$

for $k \geq 1$ and $\psi \in \Psi$. We may also assume

$$\psi(M(g(x_{\ell(k)}), g(x_{m(k)-1}))) + \psi(M(g(y_{\ell(k)}), g(y_{m(k)-1}))) \leq \varepsilon_0. \tag{2.13}$$

From (2.10) and (2.11) and by the triangle inequality and sub-additivity of ψ,

$$\begin{aligned}\varepsilon_0 \le r_k &\le \psi(M(g(x_{\ell(k)}), g(x_{m(k)-1}))) + \psi(M(g(x_{m(k)-1}), g(x_{m(k)})))\\ &+ \psi(M(g(y_{\ell(k)}), g(y_{m(k)-1}))) + \psi(M(g(y_{m(k)-1}), g(y_{m(k)})))\\ &+ \psi(M(g(x_{m(k)-1})g(x_{m(k)-1}))) + \psi(M(g(y_{m(k)-1}), g(y_{m(k)-1})))\\ &< \varepsilon_0 + \psi(M(g(x_{m(k)-1}), g(x_{m(k)}))) + \psi(M(g(y_{m(k)-1}), g(y_{m(k)})))\\ &+ \psi(M(g(x_{m(k)-1}), g(x_{m(k)-1}))) + \psi(M(g(y_{m(k)-1}), g(y_{m(k)-1}))).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.10) and (2.11) and (P2) we obtain $\lim_{k\to\infty} r_k = \varepsilon_0$.

If for some $n_0 \in \mathbb{N}$ and all $k \ge n_0$, $r_k = 0$, we have $\varepsilon_0 = 0$. Suppose $r_k > 0$ for infinitely k. Now, sub-additivity of ψ yields:

$$\begin{aligned}r_k &= \psi(M(g(x_{\ell(k)}), g(x_{m(k)}))) + \psi(M(g(y_{\ell(k)}), g(y_{m(k)})))\\ &\le \psi(M(g(x_{\ell(k)}), g(x_{\ell(k)+1})) + \psi(M(g(x_{\ell(k)+1}), g(x_{m(k)+1})))\\ &+ \psi(M(g(x_{m(k)+1}), g(x_{m(k)}))) + \psi(M(g(y_{\ell(k)}), g(y_{\ell(k)+1})))\\ &+ \psi(M(g(y_{\ell(k)+1}), g(y_{m(k)+1}))) + \psi(M(g(y_{m(k)+1}), g(y_{m(k)}))).\end{aligned}$$

$$\begin{aligned}A_k &= \psi(M(g(x_{\ell(k)}), g(x_{\ell(k)+1})) + \psi(M(g(x_{m(k)+1}), g(x_{m(k)})))\\ &+ \psi(M(g(y_{\ell(k)}), g(y_{\ell(k)+1}))) + \psi(M(g(y_{m(k)+1}), g(y_{m(k)}))).\end{aligned}$$

clearly

$$\lim_{k\to\infty} A_k = 0.$$

So

$$r_k \le A_k + \psi(M(g(x_{\ell(k)+1}), g(x_{m(k)+1}))) + \psi(M(g(y_{\ell(k)+1}), g(y_{m(k)+1}))). \tag{2.14}$$

Since, we have $g(x_{\ell(k)}) \le g(x_{m(k)})$ and $g(y_{\ell(k)}) \ge g(y_{m(k)})$ and by (3) and (6)

$$\begin{aligned}\psi(m(g(x_{\ell(k)+1}), g(x_{m(k)+1}))) &= \psi(m(f(x_{\ell(k)}, y_{\ell(k)}), f(x_{m(k)}, y_{m(k)})))\\ &\le \varphi(\frac{\psi(m(g(x_{\ell(k)}), g(x_{m(k)}))) + \psi(m(g(y_{\ell(k)}), g(y_{m(k)}))}{2})\\ &= \varphi(\frac{r_k + (\psi(m_{g(x_{\ell(k)}),g(x_{m(k)})}) + \psi(m_{g(y_{\ell(k)}),g(y_{m(k)})}))}{2}).\end{aligned}$$

Likewise, we have

$$\psi(p(g(y_{\ell(k)+1}), g(y_{m(k)+1})) \le \varphi(\frac{r_k + (\psi(m_{g(x_{\ell(k)}),g(x_{m(k)})}) + \psi(m_{g(y_{\ell(k)}),g(y_{m(k)})}))}{2}).$$

So, by (2.14)

$$r_k \le A_k + 2\varphi(\frac{r_k + (\psi(m_{g(x_{\ell(k)}),g(x_{m(k)})}) + \psi(m_{g(y_{\ell(k)}),g(y_{m(k)})}))}{2})$$

$$(2.15)$$

Letting $k \to \infty$ in (2.15), gives

$$\varepsilon_0 \leq 2 \lim_{k\to\infty} \varphi(\frac{r_k}{2}) = 2 \lim_{r_k\to\varepsilon_0^+} \varphi(\frac{r_k}{2}) < 2.\frac{\varepsilon_0}{2} = \varepsilon_0,$$

a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are M-Cauchy sequences. Since (X, m) is complete, then $m(gx_n, x) \to 0$ and $m(gy_n, y) \to 0$, as $n \to \infty$ for some $x, y \in X$ in τ_m topology. So

$$\lim_{n\to\infty}(m(gx_n, x) - m_{gx_n,x}) = 0,$$
$$\lim_{n\to\infty}(m(gy_n, y) - m_{gy_n,y}) = 0. \tag{2.16}$$

Define

$$\beta_n = \psi(m(ggx_n, ggx_n) + \psi(m(ggy_n, ggy_n),$$

for any $n \geq 0$. We show that

$$\beta_{n+1} \leq 2\varphi(\frac{\beta_n}{2}), \tag{2.17}$$

for any $n \geq 0$. We have

$$\begin{aligned}\psi(m(ggx_{n+1}, ggx_{n+1})) &= \psi(m(g(f(x_n, y_n)), g(f(x_n, y_n))))\\ &\psi(m(f(gx_n, gy_n)), f(gx_n, gy_n))))\\ &\leq \varphi(\frac{\psi(m(ggx_n, ggx_n)) + \psi(m(ggy_n, ggy_n))}{2})\\ &= \varphi(\frac{\beta_n}{2}).\end{aligned}$$

Similarly,

$$\psi(m(ggy_{n+1}, ggy_{n+1})) \leq \varphi(\frac{\beta_n}{2}).$$

Now adding two above inequalities yields (2.17). Since $\varphi(t) \leq t$ for $t \geq 0$, $\{\beta_n\}$ is monotone decreasing. Therefore $\beta_n \to \beta$ for some β. We show that $\beta = 0$. Suppose to the contrary that $\beta > 0$. Then, taking the limit of both sides of $(*)$ as $n \to \infty$, we have

$$\beta = \lim_{n\to\infty} \beta_n \leq 2 \lim_{n\to\infty} \varphi(\frac{\beta_n}{2}) = 2 \lim_{\beta_n\to\beta} \varphi(\frac{\beta_n}{2}) < 2\frac{\beta}{2} = \beta,$$

a contradiction. Therefore $\beta = 0$. Then

$$\psi(m(ggx_n, ggx_n)) \to 0 \quad \text{and} \quad \psi(m(ggy_n, ggy_n)) \to 0.$$

So,

$$m(ggx_n, ggx_n) \to 0 \quad \text{and} \quad m(ggy_n, ggy_n) \to 0. \tag{2.18}$$

Since $gx_n \to x$ and $gy_n \to y$ with m and g is m-continuous, we have $m(ggx_n, gx) \to 0$, $m(ggy_n, gy) \to 0$. Then

$$\lim_{n\to\infty}(m(ggx_n, gx) - m_{ggx_n,ggx}) = 0,$$
$$\lim_{n\to\infty}(m(ggy_n, gy) - m_{ggy_n,ggy}) = 0.$$

And also we get

$$m(gx, gx) = 0 \text{ and } m(gy, gy) = 0.$$

Case 1. Suppose f is continuous. So, $f(gx_n, gy_n) \to f(x, y))$. continuity of g, implies $g(f(x_n, y_n)) \to gx$. So by

$$f(gx_n, gy_n) = g(f(x_n, y_n)) = g(gx_{n+1}),$$

$gx = f(x, y)$. Similarly, $gy = f(y, x)$. $g(f(x_n, y_n)) \to gx$. So by

$$f(gx_n, gy) = g(f(x_n, y_n)) = g(gx_{n+1}),$$

$gx = f(x, y)$. Similarly, $gy = f(y, x)$.

Case 2. Suppose that condition (b) is satisfied.

$$\psi(m(f(x, y), f(x, y))) \leq \varphi(\frac{\psi(m(gx, gx)) + \psi(m(gy, gy))}{2}) = 0,$$

so we get

$$m(f(x, y), f(x, y)) = 0.$$

Since, $\{gx_n\}$ and $\{gy_n\}$ is non-decreasing and non-increasing respectively, and $gx_n \to x$ and $gy_n \to y$, we get $ggx_n \leq gx$ and $ggy_n \geq gy$ for any $n \geq 1$. Therefore for $\psi \in \Psi$

$$\begin{aligned}
&\psi(m(gx, f(x, y))) \\
&\leq \psi(m(gx, g(f(x_n, y_n)))) + \psi(m(f(gx_n, gy_n), f(x, y))) + \psi(m(f(gx_n, gy_n), f(gx_n, gy_n))) \\
&\leq \psi(m(gx, g(f(x_n, y_n)))) + \varphi(\frac{\psi(m(ggx_n, gx)) + \psi(m(ggy_n, gy))}{2}) + \psi(m(ggx_{n+1}, ggx_{n+1})) \\
&\leq \psi(m(gx, ggx_{n+1})) + \frac{\psi(m(ggx_n, gx)) + \psi(m(ggy_n, gy))}{2} + \psi(m(ggx_{n+1}, ggx_{n+1})),
\end{aligned}$$

for all $n \geq 1$. If in the above inequalities, $n \to \infty$, from (2.18), we have

$$\psi(m(gx, f(x, y))) \leq \psi(m(gx, gx)) + \frac{\psi(m(gx, gx)) + \psi(m(gy, gy))}{2}.$$

Because $m(gx, gx) = 0$ and $m(gy, gy) = 0$, we have $\psi(m(gx, f(x, y))) = 0$ and so $m(gx, f(x, y)) = 0$. Then $gx = f(x, y)$ and similarly $gy = f(y, x)$. □

Remark 2.3. *If in the above theorem, m is a metric and we set $\psi(t) = t$, Theorem 1.5 is reduced.*

Remark 2.4. *In Theorem 1.5, in condition (b), $x_n \leq x$ and $y \leq y_n$ are not correct and should be replaced by $gx_n \leq gx$ and $gy \leq gy_n$.*

If $(X, \leq)$ is a partially ordered set, we can endowed $X \times X$ with a partial order as follows:

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_2 \leq y_1.$$

Example 2.5. *Let $X = [0, \infty)$, $m : X \times X \to \mathbb{R}^+$ be defined by $m(x, y) = \frac{x+y}{2}$ for all $x, y \in X$, then (X, m) is an M-metric space. Define*
$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\phi(t) = \frac{1}{2}t$,
$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = t$,
$f : X \times X \to X$ *by* $f(x, y) = x + y$ *and* $g : X \to X$ *by* $g(x) = 8x$ *for all* $x, y \in X$.
We show that the condition of Theorem 2.2 holds.
We have $f(gx, gy) = gx + gy = 8x + 8y = 8(x + y) = g(x + y) = g(f(x, y))$. *So g is continuous and commute with f. On the other hand for all $x_1, x_2 \in X$ if $gx_1 \leq gx_2$, then*

$$8x_1 \leq 8x_2 \Rightarrow x_1 \leq x_2 \Rightarrow x_1 + y \leq x_2 + y \text{ for all } y \in X \Rightarrow f(x_1, y) \leq f(x_2, y).$$

Also for all $y_1, y_2 \in X$ when $gy_1 \leq gy_2$ we have

$$8y_1 \leq 8y_2 \Rightarrow y_1 \leq y_2 \Rightarrow x + y_1 \leq x + y_2 \text{ for all } x \in X \Rightarrow f(x, y_1) \leq f(x, y_2).$$

Hence f has the mixed g-monotone property. Let $(x_0, y_0) = (0, y_0)$ for any $y_0 \in X$, so we have $0 = g(0) \leq f(0, y_0) = y_0$ and $8y_0 = g(y_0) \geq y_0 = f(y_0, 0)$. And clearly the defined functions f, g, ψ, ϕ satisfied in contraction (2.1)*,also part b of theorem 2.2 hold, and we have the point $(0, 0) \in [0, \infty)$ as coupled coincidence point for f and g.*

The following uniqueness theorem is a generalization of [8, Theorem 2.2].

Theorem 2.6. *If in Theorem 2.2, we also suppose that for any (x, y) and (x^*, y^*) in $X \times X$, there is $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(x^*, y^*), f(y^*, x^*))$, then there is a unique point (z, w) such that $z = gz = f(z, w), w = gw = f(w, z)$.*

Proof. By Theorem 2.2, f and g have a coupled coincidence point (x, y). Suppose (x^*, y^*) is another coupled coincidence point for f, g. We prove

$$gx = gx^* \ , \ gy = gy^*. \tag{2.19}$$

Suppose $(u, v) \in X \times X$ is such that $(f(u, v), f(v, u))$ is comparable with $(f(x, y), f(y, x))$ and $(f(x^*, y^*), f(y^*, x^*))$. Put $u_0 = u$, $v_0 = v$. Then there are $u_1, v_1 \in X$, such that $g(u_1) = f(u_0, v_0), g(v_1) = f(v_0, u_0)$. By a similar argument as in proof of Theorem 2.2, we can construct two sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$g(u_{n+1}) = f(u_n, v_n) \quad \text{and} \quad g(v_{n+1}) = f(v_n, u_n).$$

Because, $(f(u, v), f(v, u)) = (gu_1, gv_1)$ is comparable with $(f(x, y), f(y, x)) = (gx, gy)$, with an easy way we see (gu_n, gv_n) is comparable with (gx, gy). So, by (2.1) we have

$$\psi(m(gx, gu_{n+1})) = \psi(m(f(x, y), f(u_n, v_n))) \leq \varphi(\frac{\psi(m(gx, gu_n)) + \psi(m(gy, gv_n))}{2})$$

and

$$\psi(m(gy, gv_{n+1})) = \psi(m(f(y, x), f(v_n, u_n))) \leq \varphi(\frac{\psi(m(gy, gv_n)) + \psi(m(gx, gu_n))}{2})$$

for $\psi \in \Psi$. Adding two above inequalities yields

$$\frac{\psi(m(gx, gu_{n+1})) + \psi(m(gy, gv_{n+1}))}{2} \leq \varphi(\frac{\psi(m(gx, gu_n)) + \psi(m(gy, gv_n))}{2}). \tag{2.20}$$

We set

$$\alpha_n = \frac{\psi(m(gx, gu_n)) + \psi(m(gy, gv_n))}{2}.$$

Now, by (2.20), $0 \leq \alpha_{n+1} \leq \varphi(\alpha_n) \leq \alpha_n$ for $n \geq 1$, so $\{\alpha_n\}$ is a non-increasing sequence. Then $\alpha_n \to \alpha$ for some $\alpha \geq 0$. If $\alpha > 0$, we have

$$\alpha = \lim_{n\to\infty} \alpha_{n+1} \leq \lim_{n\to\infty} \varphi(\alpha_n) = \lim_{\alpha_n \to \alpha^+} \varphi(\alpha_n) < \alpha,$$

a contradiction. Thus, $\alpha = 0$. Therefore

$$\psi(m(gx, gu_n)) \to 0 \,,\ \psi(m(gy, gv_n)) \to 0.$$

So

$$m(gx, gu_n) \to 0 \,,\ m(gy, gv_n) \to 0,$$

then by (m2)

$$m_{gx,gu_n} \to 0 \,,\ m_{gy,gv_n} \to 0.$$

Therefore we have

$$\lim_{n\to\infty} (m(gx, gu_n) - m_{gx,gu_n}) = 0, \text{ and } \lim_{n\to\infty} (m(gy, gv_n) - m_{gy,gv_n}) = 0. \tag{2.21}$$

Similarly,

$$m(gx^*, gu_n) \to 0 \,,\ m(gy^*, gv_n) \to 0,$$

and

$$m_{gx^*,gu_n} \to 0 \,,\ m_{gy^*,gv_n} \to 0.$$

Thus

$$\lim_{n\to\infty} (m(gx^*, gu_n) - m_{gx^*,gu_n}) = 0, \text{ and } \lim_{n\to\infty} (m(gy^*, gv_n) - m_{gy^*,gv_n}) = 0. \tag{2.22}$$

Now, by relations (2.21) and (2.22) we have

$$m(gx, gx) = 0, \quad \text{and} \quad m(gx^*, gx^*) = 0. \tag{2.23}$$

By (m4)

$$\psi(m(gx, gx^*) - m_{gx,gx^*}) \leq \psi(m(gx, gu_n) - m_{gx,gu_n}) + \psi(m(gu_n, gx^*) - m - gu_n, gx^*).$$

Now by (2.23) $m_{gx,gx^*} = 0$, and by (2.21) and (2.22) the right hand side of the above inequality tends to zero, then $\psi(m(gx, gx^*)) = 0$ which implies $m(gx, gx^*) = 0$, therefore, by (m1) $gx = gx^*$. Likewise, $gy = gy^*$. By commutativity of f and g,

$$\begin{aligned} g(gx) &= g(f(x, y)) = f(gx, gy), \\ g(gy) &= g(f(y, x)) = f(gy, gx). \end{aligned} \tag{2.24}$$

Therefore (gx, gy) is also a coupled coincidence point of f and g. So, by (2.19),

$$gx = ggx, gy = ggy.$$

By (2.24), $gx = f(gx, gy), gy = f(gy, gx)$.
For uniqueness of this point, suppose (z', w') is another point with

$$z' = gz' = f(z', w'), w' = gw' = f(w', z').$$

So, (gx, gy) and (z', w') are two coupled coincidence points of f and g. By (2.19), we have

$$gx = ggx = gz' = z', gy = ggy = gw' = w',$$

and the proof is complete. □

Remark 2.7. *It is noted that, the relation (31) in the proof of [5, Theorem 2.2], is not correct because in general φ is not non-decreasing.*

Conclusion

After that, Lakshmikantam and Ćirić in [5], gave some interesting fixed point results on ordered metric spaces, but there exist two error in that paper. In this paper, using a class of control functions, we not only extend and improve the main results of [5] to complete M-metric spaces but also according to Remarks 2.3 and 2.4, we mention for modifying of Theorem 1.5 in [5] as well. Even by Remark 2.7 the relation (31) in the proof of [5, Theorem 2.2], is not correct since in general φ is not non-decreasing.

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