Results in Fixed Point Theory and Applications

Existence and stability of fractional implicit differential equations with complex order

D Vivek, K Kanagarajan and E M Elsayed

Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India.

* Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

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Abstract: In this paper, we study existence results for nonlocal initial value problems for fractional implicit differential equations with complex order. The Krasnoselkii’s fixed point theorem and Banach contraction principle are used for proving the main results. Moreover, we discuss the Ulam-Hyers stability.

Keywords: Fractional implicit differential equations, Nonlocal condition, Complex order, Existence.

MSC: 26A33.

1 Introduction

The study of fractional differential equations (FDEs) ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. FDEs appear naturally in a number of fields such as physics, polymer rheology, regular variational in thermodynamics,
biophysics, electrical circuits, electron-analytical chemistry, biology, control theory, etc. An excellent description in the study of FDEs can be found in [11, 18, 19, 21]. It is considerable that there are many works about fractional implicit differential equations (FIDEs) (see, for example, [5, 6, 7]).

Ulam’s stability problem [12] has been attracted by many famous researchers, for example, see Andras, Jung and Rus [2, 14, 23]. For more recent contribution on such interesting topic, see [2, 13, 17, 22, 29, 30] and references therein.

The topics of FDEs, which attracted a growing interest for some time, in particular, in relation to the complex order in fractional calculus, have been rapidly developed recent years. E. R. Love [16] started the research of fractional derivatives of imaginary order. The concept is usual definitions of fractional integrals and derivatives by defining derivatives of purely imaginary orders. The notion of fractional operator of complex order, introduced by Samko et al.[24]. In this direction, several notions of fractional derivative of complex order were discussed [1, 25]. For instance, C.M.A.Pinto [8] introduced the two approximations of the complex order van der Pol oscillator. In the paper [20], the authors investigated the existence of solutions of boundary value problems(BVPs) with complex order. Most recently, Vivek et al. studied the existence and stability results for pantograph equations[28] and integro-differential equations[27] with nonlocal conditions involving complex order.

Motivated by the works mentioned in [6, 16, 20, 25, 27], in this paper, we consider the following nonlocal problem of fractional implicit differential equation with complex order

\[
\begin{cases}
(D^\theta_0)x(t) = f(t, x(t), D^\theta_0x(t)), & t \in J := [0, 1], \quad \theta = m + ia, \\
x(0) + g(x) = x_0,
\end{cases}
\]

where \(D^\theta_0\) is the Caputo fractional derivative of order \(\theta \in \mathbb{C}\). Let \(a \in \mathbb{R}^+, \ m \in (0, 1]\) and \(f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ g : C(J, \mathbb{R}) \to \mathbb{R}\) are given continuous functions.

It is seen that problem (1) is equivalent to the following nonlinear integral equation(see [26, 20] for more details).

\[
x(t) = x_0 - g(x) + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), D^\theta_0x(s))ds, \quad t \in [0, 1].
\]

Let \(C(J, \mathbb{R})\) be the Banach space of continuous function \(x(t)\) with \(x(t) \in \mathbb{R}\) for \(t \in J\) and \(\|x\|_\infty = \sup_{t \in J} \|x(t)\|\).

In passing, we note that the application of nonlinear condition \(x(0) + g(x) = x_0\) in physical problems yields better effect than the initial condition \(x(0) = x_0\).

The outline of the paper is as follows. In Section 2, we give some basic definitions and results concerning the complex derivative. In Section 3, we present our main results by Krasnoselkii’s fixed point theorem. In section 4, we discuss the stability results.

2 Prerequisites

In what follows we introduce definitions, notations and preliminary facts which are used in the sequel.
Theorem 2.1. [21] The Riemann-Liouville fractional integral of order $\theta \in \mathbb{C}$, $(\text{Re}(\theta) > 0)$ of a function $f : (0, \infty) \to \mathbb{R}$ is

$$I^\theta_0 f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s)ds.$$ 

Definition 2.2. [21] For a function $f$ given on the interval $J$, the Caputo fractional-order $\theta \in \mathbb{C}$, $(\text{Re}(\theta) > 0)$ of $f$, is defined by

$$(D^\theta_0 f)(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^n(s)ds,$$

where $n = \lfloor \text{Re}(\theta) \rfloor + 1$ and $\lfloor \text{Re}(\theta) \rfloor$ denotes the integral part of the real number $\theta$.

Definition 2.3. [15] The Stirling asymptotic formula of the Gamma function for $z \in \mathbb{C}$ is following

$$\Gamma(z) = (2\pi)^{1/2} z^{-1/2} e^{-z} \left[ 1 + O \left( \frac{1}{z} \right) \right], \quad (|\text{arg}(z)| < \pi; |z| \to \infty),$$

and its results for $|\Gamma(u + iv)|$, $(u, v \in \mathbb{R})$ is

$$|\Gamma(u + iv)| = (2\pi)^{1/2} |v|^{-1/2} e^{-\pi|v|/2} \left[ 1 + O \left( \frac{1}{v} \right) \right], \quad (v \to \infty).$$

For the fractional implicit differential equation with complex order (1), we adopt the definitions from Rus [23] of the Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability.

Definition 2.4. The equation (1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality

$$|D^\theta_{0+} z(t) - f(t, z(t), D^\theta_{0+} z(t))| \leq \varepsilon, \quad t \in J,$$

there exists a solution $x \in C(J, \mathbb{R})$ of equation (1) with

$$|z(t) - x(t)| \leq C_f \varepsilon, \quad t \in J.$$

Definition 2.5. The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C(J, \mathbb{R})$ of the inequality

$$|D^\theta_{0+} z(t) - f(t, z(t), D^\theta_{0+} z(t))| \leq \varepsilon, \quad t \in J,$$

there exists a solution $x \in C(J, \mathbb{R})$ of equation (1) with

$$|z(t) - x(t)| \leq \psi_f \varepsilon, \quad t \in J.$$

Definition 2.6. The equation (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality

$$|D^\theta_{0+} z(t) - f(t, z(t), D^\theta_{0+} z(t))| \leq \varphi(t) \varepsilon, \quad t \in J,$$

there exists a solution $x \in C(J, \mathbb{R})$ of equation (1) with

$$|z(t) - x(t)| \leq C_f \varphi(t) \varepsilon, \quad t \in J.$$
Definition 2.7. The equation (1) is generalized Ulam-Hyers-Rassias stable with respect to \(\varphi \in C(J, \mathbb{R})\) if there exists a real number \(C_{f, \varphi} > 0\) such that for each solution \(z \in C(J, \mathbb{R})\) of the inequality
\[
\left| D_0^\rho z(t) - f(t, z(t), D_0^\rho z(t)) \right| \leq \varphi(t), \quad t \in J, \tag{8}
\]
there exists a solution \(x \in C(J, \mathbb{R})\) of equation (1) with
\[
|z(t) - x(t)| \leq C_{f, \varphi} \varphi(t), \quad t \in J.
\]

Remark 2.8. A function \(z \in C(J, \mathbb{R})\) is a solution of the inequality (5) if and only if there exists a function \(g \in C(J, \mathbb{R})\) (which depend on \(z\)) such that
\[
(i) \quad |g(t)| \leq \epsilon, \quad t \in J;
(ii) \quad D_0^\rho z(t) = f(t, z(t), D_0^\rho z(t)) + g(t), \quad t \in J.
\]

Remark 2.9. Clearly,
\[
(i) \quad \text{Definition 2.4 } \Rightarrow \text{ Definition 2.5.}
(ii) \quad \text{Definition 2.6 } \Rightarrow \text{ Definition 2.7.}
\]

Lemma 2.10. (see Lemma 7.1.1,[12]) Let \(z, w : [0, T) \rightarrow [0, \infty)\) be continuous functions where \(T \leq \infty\). If \(w\) is nondecreasing and there are constants \(k \geq 0\) and \(0 < q < 1\) such that
\[
z(t) \leq w(t) + k \int_0^t (t - s)^{q-1} z(s) ds, \quad t \in [0, T),
\]
then
\[
z(t) \leq w(t) + \int_0^t \left( \sum_{n=1}^{\infty} \frac{(k\Gamma(q))^n}{\Gamma(nq)} (t - s)^{nq-1} w(s) \right) ds, \quad t \in [0, T).
\]

Remark 2.11. Under the hypothesis of Lemma 2.10, let \(w(t)\) be a nondecreasing function on \([0, T)\). Then we have
\[
z(t) \leq w(t) E_{q,1}(k\Gamma(q)t^q).
\]

Theorem 2.12. (Krasnoselkii fixed point theorem) Let \(K\) be a closed convex and nonempty subset of a Banach space \(X\). Let \(T\) and \(S\), be two operators such that
\[
\bullet \quad Tx + Sy \in K \text{ for any } x, y \in K;
\]
\[
\bullet \quad T \text{ is compact and continuous};
\]
\[
\bullet \quad S \text{ is contraction mapping}.
\]
Then there exists \(z_1 \in K\) such that \(z_1 = Tz_1 + Sz_1\).

3 Existence and uniqueness results

Let us list some hypotheses to prove our main results.

(A1) \(f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous function.
(A2) There exist constants \(K > 0\) and \(L > 0\) such that
\[
|f(t, u, v) - f(t, \eta, \nu)| \leq K |u - \eta| + L |v - \nu|,
\]
for any \(u, v, \eta, \nu \in \mathbb{R}\) and \(t \in J\).

(A3) \(g : C(J, \mathbb{R}) \rightarrow \mathbb{R}\) is continuous and \(b > 0\) such that
\[
|g(x) - g(y)| \leq b |x - y|, \quad \text{for all } x, y \in C(J, \mathbb{R}).
\]

(A4) There exist \(l, p, q \in C(J, \mathbb{R})\) with \(l^* = \sup_{t \in J} l(t) < 1\) such that
\[
|f(t, u, v)| \leq l(t) + p(t) |u| + q(t) |v|,
\]
for \(t \in J, u, v \in \mathbb{R}\).

(A5) There exists an increasing function \(\varphi \in C[J, \mathbb{R}]\) and there exists \(\lambda_\varphi > 0\) such that for any \(t \in J\)
\[
I^0_0 \varphi(t) \leq \lambda_\varphi \varphi(t).
\]

The main results are based on Theorem 2.12.

**Theorem 3.1.** Assume that hypotheses (A1)-(A3) are fulfilled. If \(\Omega_{b, K, L, m, \theta} < 1\), then the problem (1) has a unique solution.

**Proof.** Define the operator \(P : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})\) by
\[
(Px)(t) := x_0 - g(x) + \frac{1}{\Gamma(\theta)} \int_0^t (t - s)^{\theta - 1} f(s, x(s), D^\theta_0 x(s)) ds.
\]
It can be written as
\[
(Px)(t) = x_0 - g(x) + \left[ I^\theta_0 f(s, x(s), D^\theta_0 x(s)) \right](t).
\]
For brevity, let we take
\[
K_x(t) := D^\theta_0 x(t) = f(t, x(t), D^\theta_0 x(t)) = f(t, x(t), K_x(t)).
\]

It is clear that the fixed point of \(P\) are solutions of (1).

Let \(x, y \in C(J, \mathbb{R})\) and \(t \in J\), then we have
\[
|(Px)(t) - (Py)(t)| = |g(x) - g(y)| + \frac{1}{\Gamma(\theta)} \int_0^t (t - s)^{\theta - 1} |K_x(s) - K_y(s)| ds,
\]
and
\[
|K_x(t) - K_y(t)| \leq |f(t, x(t), K_x(t)) - f(t, y(t), K_y(t))| \
\leq K |x(t) - y(t)| + L |K_x(t) - K_y(t)| \
\leq \frac{K}{1 - L} |x(t) - y(t)|.
\]
By replacing (10) in the inequality (9), we get
\[
|(Px)(t) - (Py)(t)| \leq b|x - y| + \frac{K}{1 - L} \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |x(s) - y(s)|\, ds
\]
\[
\leq b|x - y| + \frac{K}{(1 - L) \Gamma(\theta)} \int_0^t (t-s)^{m-1} |x(s) - y(s)|\, ds
\]
\[
\leq \left( b + \frac{K}{(1 - L)m \Gamma(\theta)} \right) \|x - y\|
\]
where \( \Omega_{b,K,L,m,\theta} := \left( b + \frac{K}{(1 - L)m \Gamma(\theta)} \right) \) depends only on the parameters of the problem (1). And since \( \Omega_{b,K,L,m,\theta} < 1 \) the results follows in view of the contraction mapping principle.

**Theorem 3.2.** Assume that the hypotheses (A1), (A3) with \( b < 1 \) and (A4) hold. Then, problem (1) has at least one fixed point on \( J \).

**Proof.** Choose \( r' \geq |x_0| + G + \frac{1}{(1 - q^*) m \Gamma(\theta)} \left[ t^* + p^* r' \right] \) and consider \( B_{r'} = \{ x \in C(J, \mathbb{R}) : |x| \leq r' \} \).

Let \( A \) and \( B \) the two operators defined on \( B_{r'} \) by
\[
(Ax)(t) := \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s,x(s),D_0^\theta x(s))\, ds,
\]
and
\[
(Bx)(t) := x_0 - g(x),
\]
respectively. Note that \( x, y \in B_{r'} \), then \( Ax + By \in B_{r'} \) and \( G = \sup_{x \in C(J, \mathbb{R})} |g(x)| \). Indeed it is easy to check the inequality
\[
|Ax + By| = \left| x_0 - g(y) + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_s(s)\, ds \right|
\]
\[
\leq |x_0| + |g(y)| + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |K_s(s)|\, ds,
\]
(11)

By (A4), for each \( t \in J \), we have
\[
|K_s(t)| = |f(t,x(t),K_s(t))|
\]
\[
\leq |f(t)| + p(t) |x(t)| + q(t) |K_s(t)|
\]
\[
\leq t^* + p^* |x(t)| + q^* |K_s(t)|
\]
\[
\leq \frac{t^* + p^* |x(t)|}{1 - q^*},
\]
(12)

By replacing (12) in the inequality (11), we get
\[
|Ax + By| \leq \left| x_0 \right| + G + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{m-1} \left( \frac{t^* + p^* |x(s)|}{1 - q^*} \right)\, ds
\]
\[
\leq \left| x_0 \right| + G + \frac{1}{(1 - q^*) m \Gamma(\theta)} \left[ t^* + p^* r' \right]
\]
\[
\leq r'.
\]
Thus, $Ax + By \in B_{r}$. 

By (A3), it is also clear that $B$ is contraction mapping. Produced from continuity of $x$, the operator $(Ax)(t)$ is continuous in accordance with (A1). Also we observe that

$$|(Ax)(t)| \leq \frac{1}{|\Gamma(\theta)|} \int_{0}^{t} |(t - s)^{\theta - 1}| |K_{s}(s)| ds \leq \frac{1}{(1 - \rho^{\theta})m |\Gamma(\theta)|} \left[ |t|^{\theta} + p^{\theta}r^{\theta} \right].$$

Then $A$ is uniformly bounded on $B_{r}$. Now let’s prove that $(Ax)(t)$ is equicontinuous. Let $t_{1}, t_{2} \in J$, $t_{2} \leq t_{1}$ and $x \in B_{r}$. Using the fact $f$ is bounded on the compact set $J \times B_{r}$ (thus $\sup_{(t,x) \in J \times B_{r}} |K_{s}(t)| := C_{0} < \infty$). We will get

$$|(Ax)(t_{1}) - (Ax)(t_{2})|$$

$$\leq \left| \frac{1}{|\Gamma(\theta)|} \int_{0}^{t_{1}} (t_{1} - s)^{\theta - 1}K_{s}(s) ds - \frac{1}{|\Gamma(\theta)|} \int_{0}^{t_{2}} (t_{2} - s)^{\theta - 1}K_{s}(s) ds \right|$$

$$\leq \frac{1}{|\Gamma(\theta)|} \left| \int_{t_{2}}^{t_{1}} (t_{1} - s)^{\theta - 1}K_{s}(s) ds - \int_{0}^{t_{2}} ((t_{2} - s)^{\theta - 1} - (t_{1} - s)^{\theta - 1}) K_{s}(s) ds \right|$$

$$\leq \frac{C_{0}}{|\Gamma(\theta)|} \left| \int_{t_{2}}^{t_{1}} (t_{1} - s)^{\theta - 1} ds + \int_{0}^{t_{2}} (t_{2} - s)^{\theta - 1} - (t_{1} - s)^{\theta - 1} ds \right|$$

$$\leq \frac{C_{0}}{|\Gamma(\theta)|} \left| 2(t_{1} - t_{2})^{\theta} + t_{2}^{\theta} - t_{1}^{\theta} \right|.$$

It is easy to see that function $t^{\theta}$ is uniformly continuous on $[0,1]$. Then, $A$ is equicontinuous. So, $A(B_{r})$ is relatively compact. By the Arzela-Ascoli theorem, $A$ is compact. We now conclude the results of the theorem based on the Krasnoselkii’s fixed point. Thus, the problem (1) has at least one fixed point on $J$. 

\section{Stability analysis}

In this section, we discuss the stability results for problem (1). The arguments are based on the Banach contraction principle.

\textbf{Theorem 4.1.} \textit{Let conditions (A1)-(A3) and $\Omega_{\theta, \kappa, m, \theta} < 1$ hold, then the problem (1) is Ulam-Hyers stable.}

\textbf{Proof.} Let $\epsilon$ and let $z \in C(J, \mathbb{R})$ be a function which satisfies the inequality (5) and let $x \in C(J, \mathbb{R})$ the unique solution of the following problem

$$D_{0}^{\theta} z(t) = f(t, x(t), D_{0}^{\theta} x(t)), \quad t \in J := [0,1], \theta = m + \alpha,$$

$$z(0) + g(x) = x_{0},$$

where $a \in \mathbb{R}^{+}$ and $m \in (0,1]$.

Using equation (2), we obtain

$$x(t) = x_{0} - g(x) + \left[ I_{0}^{\theta} f(s, x(s), D_{0}^{\theta} x(s)) \right](t).$$
By integration of the inequality (5) and using Remark 2.8, we get
\[ |z(t) - x_0 + g(z) - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_z(s) ds| \leq \frac{eT^m}{m|\Gamma(\theta)|}. \]
We have
\[ |z(t) - x(t)| \leq |z(t) - x_0 + g(z) - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_z(s) ds| + eT^m + b |z(t) - x(t)| + \int_0^t (t-s)^{\theta-1} |z(s) - x(s)| ds.
\]
Thus,
\[ |z(t) - x(t)| \leq \frac{eT^m}{m(1-b)m|\Gamma(\theta)|} + \frac{K}{(1-b)(1-L)|\Gamma(\theta)|} \int_0^t (t-s)^{\theta-1} |z(s) - x(s)| ds. \]
Using Lemma 2.10(Gronwall inequality) and Remark 2.11, we obtain
\[ |z(t) - x(t)| \leq \frac{eT^m}{m(1-b)m|\Gamma(\theta)|} E_{m,1}\left( \frac{K}{(1-b)(1-L)|\Gamma(\theta)|} \right). \]
Thus, the problem (1) is Ulam-Hyers stable. If we set \( \psi(\epsilon) = C_{f,\theta}; \psi(0) = 0 \), then the problem (1) is generalized Ulam-Hyers stable.

**Theorem 4.2.** Let conditions (A1)-(A3), (A5) and \( \Omega_{k,\theta} < 1 \) hold. Then, the problem (1) is generalized Ulam-Hyers-Rassias stable.

**Proof.** Let \( z \in C([,\mathbb{R}) \) be solution of the inequality (7) and let \( x \in C([,\mathbb{R}) \) the unique solution of the following problem
\[ D^{\theta}_0 x(t) = f(t, x(t), D^{\theta}_0 x(t)), \quad t \in I := [0, 1], \quad \theta = m + i\alpha, \]
\[ z(0) + g(x) = x_0, \]
where \( \alpha \in \mathbb{R}^+ \) and \( m \in (0, 1] \).
Using equation (2), we obtain
\[ x(t) = x_0 - g(x) + \left[ e^{\theta t} f(t, x(t), D^{\theta}_0 x(s)) \right] (t). \]
By integration of the inequality (7), we obtain
\[ |z(t) - x_0 + g(z) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) ds| \leq e\lambda \psi(\epsilon). \]
On the other hand, we have

\[ |z(t) - x(t)| \]
\[ \leq |z(t) - x_0 + g(z) + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1}Kds| + |g(x) - g(z) + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1}K_zds| \]
\[ \leq \varepsilon \lambda_1 \varphi(t) + |g(z) - g(x)| + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} |K_z(s) - K_x(s)||ds| \]
\[ \leq \varepsilon \lambda_1 \varphi(t) + b|z(t) - x(t)| + \frac{K}{(1-L)|\Gamma(\theta)|} \int_0^t (t-s)^{\theta-1} |z(s) - x(s)||ds|. \]

Thus,

\[ |z(s) - x(s)| \leq \varepsilon \lambda_1 \varphi(t) + \frac{K}{(1-b)(1-L)|\Gamma(\theta)|} \int_0^t (t-s)^{\theta-1} |z(s) - x(s)||ds. \]

Using Lemma 2.10(Gronwall inequality) and Remark 2.11, we obtain

\[ |z(t) - x(t)| \leq \varepsilon \lambda_1 \varphi(t) \mathbb{L}_{m,1} \left( \frac{K}{(1-b)(1-L)|\Gamma(\theta)|} \Gamma(m) \right). \]

Thus the problem (1) is generalized Ulam-Hyers-Rassias stable.

\[ \square \]

5 An Example

Consider the following problem

\[ D_{0+}^\theta x(t) = \frac{1}{2e^{\alpha+1}(1 + |x(t)| + |D_{0+}^\theta x(t)|}, \text{ for each } t \in [0, 1], \]

\[ x(0) + \sum_{i=1}^n s_i x(t_i) = 1, \]  \hspace{1cm} (13)

where \(0 < t_1 < t_2 < \ldots < t_n < 1\) and \(s_i = 1, \ldots, n\) are positive constants with

\[ \sum_{i=1}^n s_i \leq \frac{1}{3} \]  \hspace{1cm} (14)

Set

\[ f(t, u, v) = \frac{1}{2e^{\alpha+1}(1 + |u| + |v|)}, t \in J, u, v \in R. \]

Clearly, the function \(f\) is jointly continuous. For any \(u, v, \Pi, \overline{\Pi} \in R\) and \(t \in J\),

\[ |f(t, u, v) - f(t, \Pi, \overline{\Pi})| \leq \frac{1}{2e} (|u - \Pi| + |v - \overline{\Pi}|). \]

Hence condition (A2) is satisfied with \(K = L = \frac{1}{2e}\).

On the other hand, we have

\[ |g(u) - g(\Pi)| = \left| \sum_{i=1}^n s_i u - \sum_{i=1}^n \Pi \right| \]
\[ \leq \sum_{i=1}^n |u - \Pi| \]
\[ \leq \frac{1}{3} |u - \Pi|. \]
Hence, the condition (A3) is satisfied with $b = \frac{1}{4}$. We shall check that condition (7) is satisfied for suitable values of $\alpha = 1$, $m = \frac{1}{2}$ and $T = 1$. Indeed, the condition (7) is also satisfied. It follows from Theorem 3.1 that the problem (13)-(14) has a unique solution and by Theorem 4.1, the problem (13)-(14) is Ulam-Hyers stable.

References


