Layered Compression-Expansion Fixed Point Theorem

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Abstract: The layered compression-expansion fixed point theorem is an alternative approach to the Krasnoselskii fixed point theorem for perturbed operators. The layered compression-expansion fixed point theorem is used to verify the existence of a fixed point to an operator of the form $T = R + S$ (sum of operators) by verifying the existence of a fixed point for the operator defined by $M(r, s) = (R(r + s), S(r + s))$. This result is extended to the sum of $k$ operators. Moreover, an example illustrating this technique applied to a conformable right focal boundary value problem is provided.

Keywords: Fixed-point theorems, cross product, alternate inversion, compression-expansion, layered, sum of operators, conformable derivative, right focal problem.

MSC: 47H10.
1 Introduction

Krasnoselskii’s fixed point theorem for perturbed operators [7] gives conditions for the existence of a solution to an equation of the form

\[ Cx + Dx = x, \]

where \( C \) is assumed to be \( k \)-contractive and \( D \) is assumed to be completely continuous. The arguments presented in Krasnoselskii’s seminal work, like those that have generalized and extended this work (for example, see [2, 3, 5, 9, 10, 11, 12] for some extensions and generalizations), have used iterative techniques applying the \( k \)-contractive hypothesis. In this paper we look at an alternative approach to verifying that a sum of operators \( Rx + Sx \) has a fixed point, where we will utilize compression-expansion like arguments for the operators \( R \) and \( S \) which are only assumed to be completely continuous.

Note that we are not trying to verify the existence of fixed points for the operators \( R \) and \( S \); instead, we are looking for conditions on \( R \) and \( S \) that will guarantee the existence of a fixed point for the operator \( T = R + S \), which is the sum of the operators \( R \) and \( S \). Our method hinges on the ability to verify the existence of \( r^* \) and \( s^* \) that have the property that

\[ R(r^* + s^*) = r^* \quad \text{and} \quad S(r^* + s^*) = s^*. \]

We will then have that \( x^* = r^* + s^* \) is a fixed point of \( T \), since

\[ Tx^* = Rx^* + Sx^* = R(r^* + s^*) + S(r^* + s^*) = r^* + s^* = x^*. \]

To verify the existence of \( r^* \) and \( s^* \) with these properties, we will give conditions for the existence of a solution to the equation

\[ (R(r + s), S(r + s)) = (r, s) \]

by verifying the existence of a fixed point for the operator \( M \) defined on a cross product by

\[ M(r, s) \equiv (R(r + s), S(r + s)). \]

Our initial interest in this problem was to investigate the existence of fixed points of an operator \( T \) given by

\[ Tx(t) = \int_0^1 G(t, \tau) f(x(\tau)) \, d\tau, \]

where \( f \) is a nonnegative function defined on \([0, \infty)\) that can be written as \( f = f_1 + f_2 \) (the sum of monotonic functions). Hence, our operator \( T \) can be written in the form

\[ T = R + S, \]

where

\[ Rx(t) = \int_0^1 G(t, \tau) f_1(x(\tau)) \, d\tau \quad \text{and} \quad Sx(t) = \int_0^1 G(t, \tau) f_2(x(\tau)) \, d\tau. \]
There are many other types of problems where one could take advantage of the inherent structure of the operator $T$ by writing it as a sum of operators $R$ and $S$. We conclude with an example of the Layered Compression-Expansion Fixed Point Theorem to a problem of this type, namely to a conformable right focal boundary value problem in the last section.

Our main results hinge on the fixed point index and the definitions for concave and convex functionals. See the preliminary section in [1] for a summary of the definitions and theorems used in the results that follow.

2 Main Results

Note that if $P$ is a cone in a Banach Space $E$ with norm $\| \cdot \|_E$, then $P \times P$ is a cone in the Banach space $E \times E$ with norm $\|(r,s)\|_{E \times E} = \max\{\|r\|_E,\|s\|_E\}$ and the standard product topology.

Theorem 2.1. Suppose $P$ is a cone in a real Banach space $E$, $R : P \to P$ and $S : P \to P$ are operators, and $T = R + S$. If $M : P \times P \to P \times P$ defined by $M(r,s) = (R(r + s), S(r + s))$ has a fixed point $(r^*, s^*)$, then $x^* = r^* + s^*$ is a fixed point of $T$.

Proof. Suppose $P$ is a cone in a real Banach space $E$, $R : P \to P$ and $S : P \to P$ are operators, $T = R + S$, $M : P \times P \to P \times P$ is defined by $M(r,s) = (R(r + s), S(r + s))$, $r^*, s^* \in P$ and $M(r^*, s^*) = (r^*, s^*)$. Let $x^* = r^* + s^*$. Since

$$(R(r^* + s^*), S(r^* + s^*)) = M(r^*, s^*) = (r^*, s^*),$$

we have that

$$(R(r^* + s^*), S(r^* + s^*)) = M(r^*, s^*) = (r^*, s^*),$$

Hence,

$$T(x^*) = T(r^* + s^*) = R(r^* + s^*) + S(r^* + s^*) = r^* + s^* = x^*,$$

and therefore $x^*$ is a fixed point of the operator $T$. \hfill \Box

We refer to the following fixed point theorem as the Layered Compression-Expansion Fixed Point Theorem.

Theorem 2.2. Suppose $P$ is a cone in a real Banach space $E$, $\alpha$ and $\psi$ are nonnegative continuous concave functionals on $P$, $\beta$ and $\theta$ are nonnegative continuous convex functionals on $P$, and $R, S, T$ are completely continuous operators on $P$ with $T = R + S$. If there exist nonnegative numbers $a, b, c, d$ and $(r_0, s_0) \in P(\beta, b, a) \times P(\theta, c, \psi, d)$ such that:

$$(A0) \quad P(\beta, b, a) \times P(\theta, c, \psi, d) \text{ is bounded;}$$
(A1) if \( r \in \partial P(\beta, b, a, a) \) with \( \alpha(r) = a \) and \( s \in P(\theta, c, \psi, d) \),
then \( \alpha(R(r + s)) > a \);
(A2) if \( r \in \partial P(\beta, b, a, a) \) with \( \beta(r) = b \) and \( s \in P(\theta, c, \psi, d) \),
then \( \beta(R(r + s)) < b \);
(A3) if \( s \in \partial P(\theta, c, \psi, d) \) with \( \theta(s) = c \) and \( r \in P(\beta, b, a, a) \),
then \( \theta(S(r + s)) < c \);
(A4) if \( s \in \partial P(\theta, c, \psi, d) \) with \( \psi(s) = d \) and \( r \in P(\beta, b, a, a) \),
then \( \psi(S(r + s)) > d \);
then there exists an \((r^*, s^*) \in P(\beta, b, a, a) \times P(\theta, c, \psi, d)\) such that \( x^* = r^* + s^* \) is a fixed point for \( T \).

**Proof.** Since \( P \times P \) is a closed, convex subset of the Banach Space \( E \times E \), by a corollary of Dugundji’s Theorem (see [4, p 44]) we have that \( P \times P \) is a retract of the Banach space \( E \times E \). Let \( M : P(\beta, b, a, a) \times P(\theta, c, \psi, d) \to P \times P \) be defined by

\[
M(r, s) = (R(r + s), S(r + s)),
\]
and let \( H : [0, 1] \times P(\beta, b, a, a) \times P(\theta, c, \psi, d) \to P \times P \) be defined by

\[
H(t, (r, s)) = (1 - t)M(r, s) + t(r_0, s_0).
\]

Clearly, \( H \) is continuous and \( H([0, 1] \times P(\beta, b, a, a) \times P(\theta, c, \psi, d)) \) is precompact since \( M \) is continuous and \( M(P(\beta, b, a, a) \times P(\theta, c, \psi, d)) \) is precompact.

Claim: \( H(t, (r, s)) \neq (r, s) \) for all \( (t, (r, s)) \in [0, 1] \times \partial(P(\beta, b, a, a) \times P(\theta, c, \psi, d)) \).

Suppose not; that is, suppose there exists \((t_1, (r_1, s_1)) \in [0, 1] \times \partial(P(\beta, b, a, a) \times P(\theta, c, \psi, d))\) such that

\[
H(t_1, (r_1, s_1)) = (r_1, s_1).
\]
Therefore we have that

\[
(r_1, s_1) = (1 - t_1)M(r_1, s_1) + t_1(r_0, s_0) = ((1 - t_1)R(r_1 + s_1) + t_1r_0, (1 - t_1)S(r_1 + s_1) + t_1s_0),
\]
hence we have that

\[
r_1 = (1 - t_1)R(r_1 + s_1) + t_1r_0 \quad \text{and} \quad s_1 = (1 - t_1)S(r_1 + s_1) + t_1s_0.
\]
Since \((r_1, s_1) \in \partial(P(\beta, b, a, a) \times P(\theta, c, \psi, d))\), we have either

\[
r_1 \in \partial P(\beta, b, a, a) \quad \text{or} \quad s_1 \in \partial P(\theta, c, \psi, d).
\]

Case 1: \( r_1 \in \partial P(\beta, b, a, a) \)

Since \( r_1 \in \partial P(\beta, b, a, a) \) we either have

\[
a(r_1) = a \quad \text{or} \quad \beta(r_1) = b.
\]
Subcase 1.1: $a(r_1) = a$.

By condition (A1) we have $a(R(r_1 + s_1)) > a$, thus since $r_1 = (1 - t_1)R(r_1 + s_1) + t_1r_0$ it follows that by the concavity of $a$ that

\[
a = a(r_1) = a((1 - t_1)R(r_1 + s_1) + t_1r_0) \geq (1 - t_1)a(R(r_1 + s_1)) + t_1a(r_0) > a,
\]

which is a contradiction.

Subcase 1.2: $\beta(r_1) = b$.

By condition (A2) we have $\beta(R(r_1 + s_1)) < b$, thus since $r_1 = (1 - t_1)R(r_1 + s_1) + t_1r_0$ it follows that by the convexity of $\beta$ that

\[
b = \beta(r_1) = \beta((1 - t_1)R(r_1 + s_1) + t_1r_0) \leq (1 - t_1)\beta(R(r_1 + s_1)) + t_1\beta(r_0) < b,
\]

which is a contradiction.

Therefore, we have shown that $r_1 \notin \partial P(\beta, b, a, a)$.

Case 2: $s_1 \in \partial P(\theta, c, \psi, d)$.

Since $s_1 \in \partial P(\theta, c, \psi, d)$ we either have

\[
\theta(s_1) = c \text{ or } \psi(s_1) = d.
\]

Subcase 2.1: $\theta(s_1) = c$.

By condition (A3) we have $\theta(S(r_1 + s_1)) < c$, thus since $s_1 = (1 - t_1)S(r_1 + s_1) + t_1s_0$ it follows that by the convexity of $\theta$ that

\[
c = \theta(s_1) = \theta((1 - t_1)S(r_1 + s_1) + t_1s_0) \leq (1 - t_1)\theta(S(r_1 + s_1)) + t_1\theta(s_0) < c,
\]

which is a contradiction.

Subcase 2.2: $\psi(s_1) = d$.

By condition (A4) we have $\psi(S(r_1 + s_1)) > d$, thus since $s_1 = (1 - t_1)S(r_1 + s_1) + t_1s_0$ it follows that by the concavity of $\psi$ that

\[
d = \psi(s_1) = \psi((1 - t_1)S(r_1 + s_1) + t_1s_0) \geq (1 - t_1)\psi(S(r_1 + s_1)) + t_1\psi(s_0) > d,
\]

which is a contradiction.

Therefore, we have shown that $s_1 \notin \partial P(\theta, c, \psi, d)$.

Hence, we have shown that $H(t, (r, s)) \neq (r, s)$ for all $(t, (r, s)) \in [0, 1] \times \partial (P(\beta, b, a, a) \times P(\theta, c, \psi, d))$. Note, this also shows that $M(t, (r, s)) \neq (r, s)$ for all $(r, s) \in \partial (P(\beta, b, a, a) \times P(\theta, c, \psi, d))$ by letting $t = 0$ in the previous argument. Thus by the homotopy invariance property of the fixed point index

\[
i(M, P(\beta, b, a, a) \times P(\theta, c, \psi, d), P \times P) = i((r_0, s_0), P(\beta, b, a, a) \times P(\theta, c, \psi, d), P \times P),
\]
and by the normality property of the fixed point index
\[ i(M, P(\beta, b, a, a) \times P(\theta, c, \psi, d), P \times P) = i((r_0, s_0), P(\beta, b, a, a) \times P(\theta, c, \psi, d), P \times P) = 1. \]
Consequently by the solution property of the fixed point index, \( M \) has a fixed point \((r^*, s^*) \in P(\beta, b, a, a) \times P(\theta, c, \psi, d)\) and thus by Theorem 2.1, \( T = R + S \) has a fixed point
\[ x^* = r^* + s^* \in P(\beta, b, a, a) + P(\theta, c, \psi, d). \]
This completes the proof. \( \square \)

**Remark 2.3.** Although \( \theta + \beta \) is a convex functional on \( P \) and \( \psi + \alpha \) is a concave functional on \( P \), it is not the case that \( P(\beta, b, a, a) + P(\theta, c, \psi, d) \) is a subset of \( P(\theta + \beta, b + c, \psi + \alpha, a + d) \).

This is because if \( r \in P(\beta, b, a, a) \) and \( s \in P(\theta, c, \psi, d) \), then we have that
\[ \theta(r) + \beta(s) < b + c; \]
however, that does not imply that
\[ \theta(r + s) + \beta(r + s) \text{ is less than } b + c. \]

The layered compression-expansion theorem is a major shift from a standard functional fixed point theorem being applied on the set \( P(\theta + \beta, b + c, \psi + \alpha, a + d) \). The compression-expansion arguments are made on the operators \( R \) and \( S \) instead of \( T \) and the underlying sets themselves are different, since \( P(\beta, b, a, a) + P(\theta, c, \psi, d) \) is not the same as \( P(\theta + \beta, b + c, \psi + \alpha, a + d) \). We have altered both the sets and the operators that we are applying the compression-expansion arguments to when applying the layered compression-expansion theorem.

Following the same techniques as presented in Theorem 2.1 and Theorem 2.2, we can extend these notions to layering \( k \) compression-expansion arguments, as we state without proof below.

**Theorem 2.4.** Suppose \( P \) is a cone in a real Banach space \( E \), \( k \) is a positive integer, \( R_j : P \to P \) for \( j = 1, 2, \ldots, k \) are operators and \( T = \sum_{j=1}^{k} R_j \). If
\[ M : P^k \to P^k \]
is defined by
\[ M(r_1, r_2, \ldots, r_k) = \left( R_1 \left( \sum_{j=1}^{k} r_j \right), R_2 \left( \sum_{j=1}^{k} r_j \right), \ldots, R_k \left( \sum_{j=1}^{k} r_j \right) \right), \]
has a fixed point \((r_1^*, r_2^*, \ldots, r_k^*) \), then \( x^* = \sum_{j=1}^{k} r_j^* \) is a fixed point of \( T \).
Theorem 2.5. Suppose $P$ is a cone in a real Banach space $E$, $k$ is a positive integer, $a_j$ are nonnegative continuous concave functionals on $P$ for $j = 1, 2, \cdots, k$, $\beta_j$ are nonnegative continuous convex functionals on $P$ for $j = 1, 2, \cdots, k$, and $T = \sum_{j=1}^{k} R_j$. If there exist nonnegative numbers $a_j$ and $b_j$ for $j = 1, 2, \cdots, k$, $(r_{1,0}, r_{2,0}, \cdots, r_{k,0}) \in \prod_{j=1}^{k} P(\beta_j, b_j, a_j)$, $\prod_{j=1}^{k} P(\beta_j, b_j, a_j, a_j)$ is bounded and for each $j = 1, 2, \cdots, k$:

$$(A_1_j) \quad \text{if } r_j \in \partial P(\beta_j, b_j, a_j, a_j) \text{ with } a_j(r_j) = a_j \text{ and } (r_1, r_2, \cdots, r_k) \in \prod_{j=1}^{k} P(\beta_j, b_j, a_j, a_j), \text{ then } a_j(R(\sum_{j=1}^{k} r_j)) > a_j,$$

$$(A_2_j) \quad \text{if } r_j \in \partial P(\beta_j, b_j, a_j, a_j) \text{ with } \beta_j(r_j) = b_j \text{ and } (r_1, r_2, \cdots, r_k) \in \prod_{j=1}^{k} P(\beta_j, b_j, a_j, a_j), \text{ then } \beta_j(R(\sum_{j=1}^{k} r_j)) < b_j,$$

then $T$ has a fixed point $x^* \in \sum_{j=1}^{k} P(\beta_j, b_j, a_j, a_j)$.

3 Application

In this section, we will verify the existence of a positive solution applying Theorem 2.2 to the conformable right focal boundary value problem

$$D^\nu D^\nu x(t) + f(x(t)) = 0, \quad t \in (0, 1), \quad \nu \in (0, 1],$$

$$x(0) = 0 = D^\nu x(1),$$

where $f : \mathbb{R} \rightarrow [0, \infty)$ is continuous and

$$f = f_1 + f_1$$

such that $f_1 : \mathbb{R} \rightarrow [0, \infty)$ is an increasing, continuous function and $f_1 : \mathbb{R} \rightarrow [0, \infty)$ is a decreasing, continuous function. Here we use the conformable derivative, introduced by Khalil et al [6], given by

$$D^\nu f(t) := \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\nu}) - f(t)}{\epsilon}, \quad D^\nu f(0) = \lim_{\nu \to 0^+} D^\nu f(t);$$

note that if $f$ is differentiable, then

$$D^\nu f(t) = t^{1-\nu} f'(t),$$

where $f'(t) = \lim_{\epsilon \to 0} [f(t + \epsilon) - f(t)]/\epsilon$. This conformable derivative has been used recently to model a certain flow in porous media [13], and in modeling anomalous diffusion [15]. As $\nu \to 1$, the conformable boundary value problem (3.1), (3.2) becomes the classic right focal bvp

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1),$$

$$x(0) = 0 = x'(1).$$

If $x$ is a fixed point of the operator

$$Tx(t) := \int_{0}^{1} G(t, \tau) f(x(\tau)) \tau^{\nu-1} d\tau,$$

where

$$G(t, \tau) = \frac{1}{\nu} \min\{t^{\nu}, \tau^{\nu}\}, \quad (t, \tau) \in [0,1] \times [0,1], \quad \nu \in (0,1],$$
then \( x \) is a solution of the boundary value problem (3.1), (3.2).

Let \( E = C[0,1] \) and \( P \) be the subset of nonnegative, nondecreasing, concave elements of \( C[0,1] \) then we have that \( P \) is a cone in the Banach Space \( E \) where we will use the sup-norm.

For \( x\in P \), define the convex functionals \( \beta \) and \( \theta \) on \( P \) by

\[
\beta(x) := \max_{t\in[0,1]} x(t) = x(1) \quad \text{and} \quad \theta(x) := \max_{t\in[0,1/4]} x(t) = x\left(\frac{1}{4}\right),
\]

and the concave functionals \( a \) and \( \psi \) on \( P \) by

\[
a(x) := \min_{t\in[1/4]} x(t) = x\left(\frac{1}{4}\right) \quad \text{and} \quad \psi(x) := x(1).
\]

We are now ready to prove the existence of a positive solution to the conformable right focal bvp (3.1), (3.2), by applying the Layered Compression-Expansion Fixed Point Theorem (Theorem 2.2).

**Theorem 3.1.** Let \( \nu \in (0,1] \) and let \( a, b, c, d \) be real numbers with \( b > a \) and \( 4^\nu c > d \). If \( f, f_\dagger : [0, \infty) \rightarrow [0, \infty) \) are continuous with

\[
(a) \quad f_\dagger(a + d/4) > \frac{\nu^2 16 c}{4 - \nu},
\]

\[
(b) \quad f_\dagger(b + 4c) < 2\nu^2 b,
\]

\[
(c) \quad f_\dagger(a + d/4) > \frac{\nu^2 16 d}{4 - \nu},
\]

\[
(d) \quad f_\dagger(0) < \frac{\nu^2 16 c}{4 - \nu},
\]

then the conformable right focal problem (3.1), (3.2) has a solution \( x^* \in P \).

**Proof.** For \( x \in P \) define \( R \) and \( S \) on \( P \) by

\[
Rx = \int_0^1 G(t,s)f_\dagger(x(s)) t^{\nu-1} \, ds \quad \text{and} \quad Sx = \int_0^1 G(t,s)f_\dagger(x(s)) t^{\nu-1} \, ds.
\]

Clearly \( R \) and \( S \) are completely continuous operators on \( P \) by properties of the Green’s function and applying an Arzela-Ascoli Theorem argument. Also, if \( r \in \overline{P(\beta, b, a, a)} \) then \( \|r\| \leq b \) and if \( s \in \overline{P(\theta, c, \psi, d)} \) then \( \theta(s) = s\left(\frac{1}{4}\right) \leq c \) hence by the concavity of \( s \) we have \( 4s\left(\frac{1}{4}\right) \geq s(1) \) hence \( \|s\| = s(1) \leq 4c \). Therefore, \( P(\beta, b, a, a) \times P(\theta, c, \psi, d) \) is a bounded, open subset of \( P \times P \).

Let \( r_1 \in \partial P(\beta, b, a, a) \) and \( s_1 \in \overline{P(\theta, c, \psi, d)} \).

Claim 1: If \( a(r_1) = a \), then \( a(R(r_1 + s_1)) > a \).

Since \( r_1 \) and \( s_1 \) are increasing, for all \( \tau \geq \frac{1}{4} \) we have that

\[
r_1(\tau) + s_1(\tau) \geq r_1\left(\frac{1}{4}\right) + s_1\left(\frac{1}{4}\right) \geq a + \frac{d}{4},
\]

thus by (a) using that \( f_\dagger \) is increasing we have that
\[
\alpha(R(r_1 + s_1)) = \int_0^1 G \left( \frac{1}{4} \tau \right) f_\tau(r_1(\tau) + s_1(\tau)) \tau^{\nu - 1} \, d\tau \\
\geq \int_1^4 G \left( \frac{1}{4} \tau \right) f_\tau(r_1(\tau) + s_1(\tau)) \tau^{\nu - 1} \, d\tau \\
= \int_1^4 \frac{f_\tau(r_1(\tau) + s_1(\tau))}{\nu^{\nu \nu - 1}} \tau^{\nu - 1} \, d\tau \\
\geq \frac{(4^\nu - 1)f_\tau \left( a + \frac{d}{4} \right)}{\nu^{2\nu 16^\nu}} > a.
\]

Claim 2: If \( \beta(r_1) = b \), then \( \beta(R(r_1 + s_1)) < b \).

Since \( r_1 \) and \( s_1 \) are concave and increasing, for all \( \tau \in [0, 1] \) we have that
\[
r_1(\tau) + s_1(\tau) \leq r_1(1) + s_1(1) \leq b + 4c,
\]
thus by (b) using that \( f_\tau \) is increasing we have that
\[
\beta(R(r_1 + s_1)) = \int_0^1 G(1, \tau) f_\tau(r_1(\tau) + s_1(\tau)) \tau^{\nu - 1} \, d\tau \\
= \int_0^1 \frac{\tau^{\nu}}{\nu^{\nu \nu}} f_\tau(r_1(\tau) + s_1(\tau)) \tau^{\nu - 1} \, d\tau \\
\leq \frac{f_\tau \left( b + 4c \right)}{2\nu^2} < b.
\]

Let \( s_2 \in \partial P(\theta, c, \psi, d) \) and \( r_2 \in \overline{P(b, a, a)} \).

Claim 3: If \( \psi(s_2) = d \), then \( \psi(S(r_2 + s_2)) > d \).

Since \( r_2 \) and \( s_2 \) are increasing, for all \( \tau \geq \frac{1}{4} \) we have that
\[
r_2(\tau) + s_2(\tau) \geq r_2 \left( \frac{1}{4} \right) + s_2 \left( \frac{1}{4} \right) \geq a + \frac{d}{4},
\]
thus by (c) using that \( f_\tau \) is decreasing we have that
\[
\psi(S(r_2 + s_2)) = \int_0^1 G(1, \tau) f_\tau(r_2(\tau) + s_2(\tau)) \tau^{\nu - 1} \, d\tau \\
\geq \int_{\frac{1}{4}}^1 \frac{\tau^{\nu}}{\nu^{\nu \nu}} f_\tau(r_2(\tau) + s_2(\tau)) \tau^{\nu - 1} \, d\tau \\
\geq \frac{1}{\nu} \int_{\frac{1}{4}}^1 f_\tau \left( a + \frac{d}{4} \right) \tau^{\nu - 1} \, d\tau \\
= \frac{(16^\nu - 1)f_\tau \left( a + \frac{d}{4} \right)}{2\nu^2 16^\nu} > d.
\]

Claim 4: If \( \theta(s_2) = c \), then \( \theta(S(r_2 + s_2)) < c \).
We have that \( r_2 \) and \( s_2 \) are nonnegative, thus by (d) using that \( f_j \) is decreasing we have that

\[
\theta(S(r_2 + s_2)) = \int_0^1 G\left(\frac{1}{4}, \tau\right) f_j(r_2(\tau) + s_2(\tau)) \tau^{v-1} \, d\tau
\]

\[
= \int_0^1 G\left(\frac{1}{4}, \tau\right) f_j(r_2(\tau) + s_2(\tau)) \tau^{v-1} \, d\tau
\]

\[
+ \int_1^4 G\left(\frac{1}{4}, \tau\right) f_j(r_2(\tau) + s_2(\tau)) \tau^{v-1} \, d\tau
\]

\[
\leq \int_0^1 f_j(0) \frac{\tau^v}{v} \tau^{v-1} \, d\tau + \int_1^4 f_j(0) \frac{\tau^v}{v} \tau^{v-1} \, d\tau
\]

\[
= \frac{f_j(0)}{2\nu^v} + \frac{(4^v - 1) f_j(0)}{v^v}
\]

\[
\leq \frac{(4^v - 1/2) f_j(0)}{v^v} < c.
\]

Finally, let

\[
s_0(t) \equiv \left(\frac{4^v c + d}{2}\right) t^v \quad \text{and} \quad r_0(t) \equiv \begin{cases} \frac{1}{2} (4t)^v (b + a) & \text{if } t \leq \frac{1}{4}, \\ \frac{b + a}{2} & \text{if } t \geq \frac{1}{4}. \end{cases}
\]

Then \( (r_0, s_0) \in P(\beta, b, a) \times P(\theta, c, \psi, d) \) and the hypotheses of Theorem 2.2 are met. Consequently, \( T \) has a fixed point

\[ x^* = r^* + s^* \in P(\beta, b, a) + P(\theta, c, \psi, d). \]