Let $X$ be set. A symmetric on $X$ is a mapping $r : X \times X \rightarrow [0, \infty)$ such that

$$r(x,y) = 0 \text{ iff } x = y, \text{ and } r(x,y) = r(y,x) \text{ for } x,y \in X.$$ 

Let $f, g$ be two selfmaps of a metric space $X$. $f$ and $g$ are said to be occasionally weakly compatible (owc) if there exists a common coincidence point $w$ of $X$ at which $f$ and $g$ commute.

The following is Theorem 1 of [6].

**Theorem 1.** Let $X$ be a set with a symmetric $r$. Suppose that $f, g, S, T$ are selfmaps of $X$, and that the pairs \{f, S\} and \{g, T\} are each owc. If

$$r(fx, gy) < M(x,y)$$ 

(1)

for each $x, y \in X$ for which $x \neq y$, where

$$M(x,y) = \max \{r(Sx,Ty), r(Sx,fx), r(Ty,gy), r(Sx,gy), r(Ty,fx)\},$$
then there is a unique point \( w \in X \) such that \( fw = Sw = w \) and a unique point \( z \in X \) such that \( gz = Tz = z \). Moreover, \( z = w \), so that there is a unique common fixed point of \( f, g, S, \) and \( T \).

The following is Theorem 2.1 of Chandok [4].

**Theorem 2.** Let \( M \) be a subset of a metric space \( (X,d) \). Suppose that \( T, f, g : M \rightarrow M \) satisfy

\[
d(Tx,Ty) \leq a \left( \frac{d(gx, Tx)d(gy, Ty)}{d(gx, gy) + d(gx, Ty) + d(gy, Tx)} \right) + bd(gx, gy)
\]

for all \( x, y \in M \) and for some \( a, b \in [0,1) \) with \( a + b < 1 \). Suppose also that \( T(M) \cup f(M) \subseteq g(M) \) and \( (g(M), d) \) is complete. Then \( T, f, \) and \( g \) have a unique common fixed point in \( X \).

Note that \( d(gx, Tx) \leq d(gx, gy) + d(gy, Tx) \). Therefore (2) implies that

\[
d(Tx, Ty) \leq ad(gx, fy) + bd(gx, gy) \leq (a + b) \max\{d(gx, fy), d(gx, gy)\} < \max\{d(gx, fy), d(gx, gy)\},
\]

which is a special case of (1).

Two maps are weakly compatible if they commute at every coincidence point. Therefore weakly commuting implies owc. The conclusion of Theorem 2 now follows from Theorem 1.

The following is Theorem 2.3 of [1], where \( \psi \in \Psi := \{\psi : \mathbb{R} \rightarrow \mathbb{R} : \text{ such that } \psi(t) \leq t \text{ for } t > 0 \text{ and } \psi(0) = 0 \text{ iff } t = 0\} \).

**Theorem 3.** Let \( A, B, S, \) and \( T \) be selfmaps of a complete metric space \( (X,d) \) satisfying

\[
A(X) \subseteq T(X), \quad B(X) \subseteq S(X)
\]

and

\[
d(Ax, By) \leq \psi \left( \max \left\{ \frac{d(Sx, Ax)d(Ty, By)}{d(Sx, Ty) + d(Sx, By) + d(Ty, Ax)}, d(Sx, T_{dy}) \right\} \right),
\]

for all \( x, y \in X \). Then we have the following:

(i) If \( A(X) \subseteq T(X) \) and the pair \( (B, T) \) is weakly compatible, and if \( z \) is a common fixed point of \( A \) and \( S \) then \( z \) is a common fixed point of \( A, B, S \) and \( T \) and it is unique.

(ii) If \( B(X) \subseteq S(X) \) and the pair \( (A, S) \) is weakly compatible, and if \( z \) is a common fixed point of \( B \) and \( T \) then \( z \) is a common fixed point of \( A, B, S, \) and \( T \) and it is unique.

Since \( d(Sx, Ax) \leq d(Sx, Ty) + d(Ty, AX) \), (3) implies

\[
d(Ax, By) \leq \max\{d(Ty, By), d(Sx, Ty)\},
\]

which is a special case of (1), since \( \psi \in \Psi \).
Using (i), if \((B, T)\) is weakly compatible, then it is owc. If \(z\) is a common fixed point of \(A\) and \(S\), then \(z = Az = Sz\), and \((A, S)\) is owc. The conclusion now follows from Theorem 1. Case (ii) is handled in a similar manner.

Theorems 2.5 and 2.6 of [1] are also special cases of Theorem 1.

The result of [2] was generalized by [13], and was further generalized as Theorem 2.1 of [14]. The following is Theorem 2.1 of [14].

**Theorem 4.** Let \(f\) and \(g\) be self-maps of a complete metric space \(X\) satisfying \(f(X) \subset g(X)\) and the inequality

\[
\begin{align*}
[d(fx, fy)]^2 &\leq \psi \left( \max \{d(fgx, fy) + d(fy, gx)d(fx, gy), \\
&\quad d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy) \} \right) 
\end{align*}
\]

(4)

for all \(x, y \in X\), where \(\psi \in \Psi\) is nondecreasing and upper semicontinuous. If \(g\) is continuous, and \((f, g)\) is a compatible pair, then \(f\) and \(g\) have a unique common fixed point.

Condition (4) implies that

\[d(fx, fy) \leq \max \{d(fx, gx), d(fy, gy), d(fy, gx), d(fx, gy)\},\]

which is a special case of (1).

The condition \((f, g)\) is compatible implies that \((f, g)\) are owc. The result now follows from Theorem 1.

The following is Theorem 2.3 of [1].
Theorem 6. Let \( A, B, S \) and \( T \) be selfmaps of a complete metric space \((X, d)\) satisfying \( A(X) \subseteq TX \), \( B(X) \subseteq S(X) \) and

\[
d(Ax, By) \leq \begin{cases} 
\psi \left( \max \left\{ \frac{d(Sx, Ax) + d(Ty, By)}{d(Sx, Ty) + d(Ty, Ax)} \right\} \right), & \text{if } D \neq 0, \\
0, & \text{if } D = 0,
\end{cases}
\]

(6)

for all \( x, y \in X \), where \( D = d(Sx, Ty) + d(Ty, Ax) + d(Ax, Ty) \).

If the pairs \((A, S)\) and \((B, T)\) are weakly compatible and one of the range sets \( S(X), T(X), \) and \( B(X) \) is closed, then, for any \( x_0 \in X \), the sequence \( \{y_n\} \) defined by

\[
y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

is Cauchy in \( X \) and \( \lim_n y_n = z \) (say), \( z \in X \) and \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

Note that \( d(Sx, Ax) \leq d(Sx, Ty) + d(Ty, Ax) \). Using the fact that \( \psi \in \Psi \), condition (6) implies that

\[
d(Ax, By) < \max \{d(Ty, By), d(Sx, Ty)\},
\]

which is a special case of (1).

Since \((A, S)\) and \((B, T)\) are weakly compatible, they are owc, and the conclusion follows from Theorem 1.

Theorems 2.4, 2.5, and 2.6 of [1] are proved in the same way.

The following is Theorem 3.1 of [3]

Theorem 7. Let \( X \) be a set with a symmetric \( d \). Let \( f, g \) and \( h \) be three self mappings of \((X, d)\) and \( \psi \in \Psi \) satisfying

\[
d^2(fx, gy) \leq \max \{\psi(d(hx, hy))\psi(d(hx, fx)), \\
\psi(d(hx, fy))\psi(d(hx, hy))\psi(d(hy, gy)), \\
\psi(d(hx, fx))\psi(d(hy, gy))\psi(d(hy, fx))\}
\]

(7)

for all \( x, y \in X \), and the pair \((f, h)\) or \((g, h)\) is owc. Then \( f, g \) and \( h \) have a unique common fixed point.

Inequality (7) implies that

\[
d(fx, gy) < \max \{d(hx, hy), d(hx, fx), d(hy, fy), d(hy, gy), d(hx, gy), d(hy, fx)\},
\]

(8)

which is a special case of (1).

Suppose that \((f, h)\) is owc. Then there exists a point \( z \in X \) such that \( fz = hz \) and \( fhz = hfz \). From (7),

\[
d^2(fz, gz) \leq \max \{\psi(d(hz, hz))\psi(d(hz, fz)), \\
\psi(d(hz, hfz))\psi(d(hz, gz)), \psi(d(hz, fz))\psi(d(hz, gz))\}
\]

\[
\psi(d(hz, gz))\psi(d(hz, fz)) \leq 0,
\]

and \( fz = gz \).
Now substituting $x = fz, y = gz$ into (7) one obtains
\[ d^2(f gz, g fz) \leq \max\{ \psi(d(h fz, hgz)\psi(d(h fz, f^2 z)), \]
\[ \psi(d(h fz, hgz))\psi(d(hgz, f^2 z)), \psi(d(h fz, hgz))\psi(d(hgz, g^2 z)), \]
\[ \psi(d(h fz, f^2 z))\psi(d(hgz, g^2 z)), \psi(d(h fz, g^2 z))\psi(d(hgz, f^2 z)) \]
\[ \leq 0, \]
and $(f, g)$ are owc. The conclusion now follows from Theorem 1.

Using the same argument, the assumption $f z = gz$ implies that $(f, h)$ are owc, and the conclusion again follows from Theorem 1.

The following is Theorem 1 of [9]

**Theorem 8.** Let $f, g$ be selfmaps of a complete metric space with $f$ continuous. Suppose that $f$ and $g$ commute and $g(X) \subset f(X)$. Suppose that $g$ satisfies
\[ d(gx, gy) \leq a_1 d(gx, fx) + a_2 d(gy, fy) + a_3 d(gx, fy) + a_4 d(gy, fx) + a_5 d(fx, fy), \tag{9} \]
where $a_i \geq 0, \sum_{i=1}^5 a_i < 1$. Then $f$ and $g$ have a unique common fixed point in $X$.

Since $f$ and $g$ commute, they are owc. Inequality (9) implies that
\[ d(gx, gy) \leq h \max\{d(gx, fx), d(gy, fy), d(gx, fy), d(fx, fy), d(fx, fy)\}, \]
where $h = \sum_{i=1}^5 a_i < 1$.

The conclusion follows from Theorem 1.

The following is Theorem 1 of [11].

**Theorem 9.** Let $(X, d)$ be a compact metric space, $A, T : X \to X$ such that
(i) $d(Tx, Ty) < \frac{1}{2}[d(Ax, TAx) + d(Ay, TAy)]$ for all $x, y \in X$ with $Tx \neq Ty$.
(ii) $d(Ax, Ay) \leq d(x, y)$.
(iii) $TAx = ATx$.

Then $A$ and $T$ have a unique common fixed point.

By (iii), $A$ and $T$ commute. Then $A(TA) = (AT)A$ and $A$ and $AT$ commute Also $T(TA) = T(AT) = (TA)T$, and $T$ and $TA$ commute. Therefore $(A, T), (A, TA)$, and $(T, AT)$ are owc.

Inequality (1) implies that
\[ d(Tx, Ty) < \max\{d(Ax, TAx), d(Ay, TAy)\}. \]

Thus $T, A$ and $TA$ have a unique common fixed point by Theorem 1.

Note that condition (ii) is not needed.

The following is Theorem 1 of [12]
Theorem 10. Let $S, I$ and $T, J$ be two pairs of weakly commuting mappings of a complete metric space $(X, d)$ into itself satisfying the inequality

$$[d(Sx, Ty)]^3 \leq \alpha d(Ix, Jy)d(Ix, Sx)d(Jy, Ty),$$

(10)

for all $x, y \in X$, where $\alpha \in (0, 1)$. If the range of $I$ contains the range of $T$ and the range of $J$ contains the image of $S$, and if one of $S, T, I$ and $J$ is continuous, then $S, T, I$ and $J$ have a unique common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ and $I$ and $J$ and of $T$ and $J$.

From (10),

$$[d(Sx, Ty)]^3 \leq \alpha \max\{[d(Ix, Ty)]^3, [d(Ix, Sx)]^3, [d(Jy, Ty)]^3\},$$

which implies that

$$d(Sx, Ty) \leq \alpha^{1/3} \max\{d(Ix, Sy), d(Ix, Sx), d(Jy, Ty)\},$$

a special case of (1). Since $(S, I)$ and $(T, J)$ are weakly commuting they are owc. The conclusion follows from Theorem 1.

The following is Theorem 2 of [12]

Theorem 11. Let $S, I$ and $T, J$ be two pairs of weakly commuting mappings of a complete metric space $(X, d)$ into itself satisfying the inequality

$$[d(Sx, Ty)]^2 \leq \alpha \max\{d(Ix, Sx)d(Jy, Ty), d(Ix, Ty)d(Jy, Sx),$$

$$\frac{1}{2}d(Ix, Ty)d(Jy, Ty), \frac{1}{2}d(Ix, Sx)d(Jy, Sx)\},$$

(11)

for all $x, y \in X$, where $\alpha \in (0, 1)$. If the range of $I$ contains the range of $T$ and the range of $J$ contains the range of $S$ and if one of $S, T, I$ and $J$ is continuous, then $S, T, I$ and $J$ have a unique common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ and $T$ and $I$ and of $T$ and $J$.

Inequality (11) implies

$$[d(Sx, Ty)]^2 \leq \alpha \max\{d^2(Ix, Sx), d^2(Jy, Ty), d^2(Ix, Ty), d^2(Jy, Sx)\},$$

which in turn implies that

$$d(Sx, Ty) \leq \alpha^{1/2} \max\{d(Ix, Sx), d(Jy, Ty), (Ix, Ty), d(Jy, Sx)\},$$

which is a special case of (1).

Since $(S, I)$ and $(T, J)$ are weakly commuting, they are owc, and the conclusion follows from Theorem 1.

The following is Theorem 1 of [15]

Theorem 12. Let $P, Q, T$ be selfmaps of a complete metric space $X$ such that $PT = TP, QT = TQ, P(X) \cup Q(X) \subseteq T(X)$. If $T$ is continuous and there exists an $h \in (0, 1)$ such that

$$d(Px, Qy) \leq h \max\{d(Tx, Ty), d(Px, Tx), d(Qy, Ty), \frac{1}{2}[d(Px, Ty) + d(Qy, Tx)]\},$$

(12)

for all $x, y \in X$, then $P, Q, T$ have a unique common fixed point.
Inequality (12) implies that
\[ d(Px, Qy) \leq h \max \{ d(Tx, Ty), d(Px, Tx), d(Qy, Ty), s(Px, Ty), d(Qy, Tx) \} , \]
which is a special case of (1). Since \((P, T)\) and \((Q, T)\) commute, they are owc, and the conclusion follows from Theorem 1.

The following is Theorem 2.1 of [17]

**Theorem 13.** Let \(X\) be a nonempty set with symmetric \(d\) and \(f, g, f\) and \(r\) be self-maps on \((X, d)\) satisfying any two of the following three inequalities, where \(\phi \in \psi:\)
\[
d^2(fx, gy) \leq \max \{ \phi(d(rx, ry))\phi(d(rx, fx)), \phi(d(rx, ry))\phi(d(ry, fy)), \\
\phi(d(rx, gy))\phi(d(rx, fx)) \}, \tag{13} \\
d^2(gx, hy) \leq \max \{ \phi(d(rx, ry))\phi(d(rx, gx)), \phi(d(rx, ry))\phi(d(ry, gy)), \\
\phi(d(rx, hy))\phi(d(rx, gx)) \}, \tag{14} \\
d^2(hx, fy) \leq \max \{ \phi(d(rx, ry))\phi(d(ry, hx)), \phi(d(rx, ry))\phi(d(ry, fx)), \\
\phi(d(rx, fy))\phi(d(rx, hx)) \}, \tag{15} \\
\]
for all \(x, y \in X\). If any of the pairs \((f, r), (g, r)\) or \((h, r)\) is owc, then \(f, g, h\) and \(r\) have a unique common fixed point.

If, in (13), one sets \(f = g\) and \(g = h\), then (13) becomes (14). Similarly, the substitution \(f = h, g = f\) changes (13) into (15). Therefore the assumptions (13)-(15) are equivalent.

Condition (13) implies that
\[
d^2(fx, gy) \leq \max \{ \phi^2(d(rx, ry)), \phi^2(d(rx, fx)), \phi^2(d(ry, fy)) \}, \]
or
\[
d(fx, gy) \leq \max \{ \phi(d(rx, ry)), \phi(d(rx, fx)), \phi(d(ry, gy)) \} . \]

Therefore Theorem 13 is a special case of Theorem 7.

The following is Theorem 2.1 of [7].

**Theorem 14.** Let \(f\) and \(g\) be self-maps of a G-metric space \((X, G)\) satisfying \(f(X) \subseteq g(X)\),
\[
G(fx, fy, fz) \leq \alpha \max \{ G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, fz) \} , \tag{16} \\
\]
where \(\alpha \in [0, 1/2]\), and one of \(f\) or \(g\) is continuous.

Then \(f\) and \(g\) have a unique common fixed point in \(X\), provided \(f\) and \(g\) are compatible maps.
For a discussion of the basic properties of \( G \)-metric spaces the reader may consult [10]. In [18] it was shown that, if one defines
\[
d^G(x, y) = \max\{G(x, x, y), G(x, y, z)\}
\] (17)
for any \( x, y \) in \( G \), then then \( d^G(x, y) \) becomes a metric.

Substituting \( z = x \) and then \( z = y \) into (16) one obtains
\[
G(fx, fy, fx) \leq \alpha \max\{G(fx, gy, gz), G(gx, fy, gx), G(gx, gy, gx)\}
\]
and
\[
G(fx, fy, fy) \leq \alpha \max\{G(fx, gy, gy), G(gx, fy, gy), G(gx, gy, fy)\}.
\]
Using (16) one obtains
\[
d^G(fx, fy) \leq \alpha \max\{d^G(fx, fy), d^G(gx, fy), d^G(gx, gy)\},
\]
which is a special case of (1). Since \( f \) and \( g \) are compatible, they are owc. The conclusion follows from Theorem 1.

Theorems 3.5 and 4.3 of [7] are also special cases of Theorem 1.

Define \( S = \{g : \mathbb{R}^+ \rightarrow \mathbb{R}^+\} \) such that
\begin{align*}
(g_1) & \text{ } g \text{ is nondecreasing in the 4th and 5th variables,} \\
(g_2) & \text{ } \text{If } u, v \in \mathbb{R}^+ \text{ are such that } u \leq g(v, v, u, u + v), \text{ or } u \leq g(v, u, u + v), \text{ or } u \leq g(v, u, v + v) \text{, then } u \leq hv, \text{ where } 0 < h < 1 \text{ is a constant,} \\
(g_3) & \text{ } \text{If } u \in \mathbb{R}^+ \text{ is such that } u \leq g(u, 0, u, u) \text{ or } u \leq g(0, u, 0, u) \text{ or } u \leq g(0, 0, u, u) \text{, then } u = 0.
\end{align*}

The following is Theorem 3.1 of [16].

**Theorem 15.** Let \( A, B, S \) and \( T \) be selfmappings from a complete metric space \( (X, d) \) into itself satisfying the following conditions
\[
A(X) \subseteq T(X) \quad \text{and} \quad B(X) \subseteq S(X)
\]
\[
[d(Ax, By)]^2 \leq k_1[d(Ax, Sx)d(By, Ty) + d(By, Sx)d(Ax, Ty)] + k_2[d(Ax, Sx)d(Ax, Ty) + d(By, Ty)d(By, Sx)]
\]
(18)
for all \( x, y \in X \), where \( 0 \leq k_1 + 2k_2 < 1, k_1, k_2 \geq 0 \), one of the mappings \( A, S, B, T \) of \( X \) is continuous, and the pairs \( (A, S) \) and \( (B, T) \) are compatible mappings of type \( (E) \).

Further, if the sequence \( Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots \) converges to \( z \in X \), then \( A, B, S \) and \( T \) have a unique common fixed point \( z \) in \( X \).

Theorem 15 will use Theorem 4 of [6].

Define \( S = \{g : \mathbb{R}^+ \rightarrow \mathbb{R}^+\} \) such that
\( (g_1) \) is nondecreasing in the 4th and 5th variables,
(g2) If \( u, v \in \mathbb{R}^+ \) are such that \( u \leq g(v, v, u, u + v, 0) \), or

\[
u \leq g(v, u, u, v + 0) \text{ or } v \leq g(u, u, v, u + v, 0),\]

or \( u \leq g(v, u, v, u + v) \), then \( u \leq v \).

(g3) If \( u \in \mathbb{R}^+ \) is such that \( u \leq g(u, 0, 0, u, u) \) or \( u \leq g(0, 0, u, u, u) \) or

\[
u \leq g(0, 0, u, u), \text{ then } u = 0.\]

The following is Theorem 4 of [6].

**Theorem 16.** Let \( X \) be a set, \( r \) a symmetric on \( X \). Let \( f, g, S, T \) be selfmaps of \( X \) satisfying \( f(x) \subseteq T(X), g(x) \subseteq S(X) \), and

\[
r(f(x, y)) \leq g(r(Sx, Ty), r(fx, Sx), r(gy, Ty), r(fx, Ty), r(gy, Sx))
\]

for all \( x, y \in X \), where \( g \in \mathfrak{S} \). If \( \{f, S\} \) and \( \{g, T\} \) are owc, then \( f, g, S, T \) have a unique common fixed point.

The functions defined by inequality (18) satisfy conditions (g1) – (g3). Compatible mappings of type E are owc. Therefore the result follows from Theorem 16. The following is Theorem 2.1 of [8]

**Theorem 17.** Let \( (X, d) \) be a complete Takahashi metric space with convex structure \( W \) which is continuous in the third variable. Let \( K \) be a nonempty closed subset of \( X \). Let \( \delta K \) be the boundary of \( K \) with \( \delta K \neq \emptyset \). Let the mappings \( A, B, S, T : K \rightarrow K \) be selfmaps of \( X \) and satisfy the following conditions:

(i) For any \( x, y \in K, d(Ax, By) \leq M_{r}(x, y) \), where

\[
M_{r}(x, y) = \max\{\omega_{1}[d(Sx, Ty)], \omega_{2}[d(Ax, Sx)], \omega_{3}[d(By, Ty)]\}
\]

where each \( \omega_{i} : [0, \infty) \rightarrow [0, \infty), i = 1, 2, 3, 4, 5 \) is a nondecreasing semicontinuous function from the right, such that \( \omega_{i} r < r/2 \) for \( r > 0 \) and \( \lim_{r \to \infty}[r - 2 \omega_{i}(r)] = \infty \),

(ii) \( \delta K \subseteq T(K), \delta(K) \subseteq S(K) \),

(iii) \( Sx \in \delta K \) implies \( Ax \in K, Tx \in \delta K \) implies \( Bx \in K \),

(iv) \( A(K) \cap K \subset T(K), B(K) \cap K \subset S(K) \) and

(v) \( S(K), T(K) \) are closed in \( X \).

Then there exists a coincidence point \( z \in K \) for \( A, B, S \) and \( T \). Moreover, if each of the pairs \( \{A, S\} \) and \( \{B, T\} \) is coincidentally commuting, then \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

Hypothesis (i) implies

\[
d(Ax, By) \leq \frac{1}{2} \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)\},
\]

which is a special case of (1). Since the pairs \( \{A, S\} \) and \( \{B, T\} \) are coincidentally commuting, they are owc, and the conclusion follows from Theorem 1.

**References**


