Common fixed point results in metric spaces endowed with a graph

Binayak S. Choudhury¹, Nikhilesh Metiya*² and Debashis Khatua³

1,3 Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India.
2 Department of Mathematics, Sovarani Memorial College, Jagatballavpur, Howrah-711408, West Bengal, India.

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Abstract: In this paper we establish some common fixed point results for a pair of mappings in a metric space having the additional structure of a directed graph. The mappings are assumed to satisfy certain almost G-contractions without and with rational terms. Each of our results is illustrated with example. The approach here is a blending of analytic and graph theoretic methodologies.

Keywords: Metric space; directed graph; G-contraction; common fixed point.

MSC: 47H10, 54H10, 54H25.

1 Introduction

The consideration of partial ordering in metric fixed point theory appeared first in the work of Turinici [25] who worked out some results in metrizable uniform spaces. Subsequent work was carried out in this direction by Ran and Reurings [23] where they combined the Banach contraction principle and the Knaster-Tarski fixed point theorem. Ran and Reurings also combined their result with the Schauder fixed point theorem and applied it to obtain some existence and uniqueness results to nonlinear matrix equations. Neito & Rodríguez-López ([21], [22]) extended the results of Ran and Reurings. They also applied their
results to obtain a theorem on the existence of a unique solution for periodic boundary problems relative to ordinary differential equations. Following the above works, fixed point theory rapidly developed in partially ordered metric spaces. Some prominent works on fixed point and related issues are, for instances, [5, 9, 11, 12, 13, 14, 17, 18, 19, 20].

Graphs are ordered structures which include the partial ordering as a particular case. The use of graph was introduced in metric fixed point theory by Jachymski [16]. It is in furtherance of metric fixed point theory in partially ordered spaces. A version of the Banach contraction mapping principle was established was introduced in metric fixed point theory by Jachymski [16]. It is in furtherance of metric fixed point theory in partially ordered metric spaces. Some prominent works on fixed point and related issues are, for instances, [5, 9, 11, 12, 13, 14, 17, 18, 19, 20].

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Let X be a nonempty set and Δ := \{ (x, x) : x ∈ X \}. Let G be a directed graph such that its vertex set V(G) coincides with X, that is, V(G) = X and the edge set E(G) contains all loops, that is, Δ ⊆ E(G). Assume that G has no parallel edges. We can identify G with the pair G(V(G), E(G)). By G\(^{-1}\) we denote the conversion of a graph G, that is, the graph obtained from G by reversing the directions of the edges. Thus we have

\[ V(G^{-1}) = V(G) \text{ and } E(G^{-1}) = \{ (x, y) ∈ X × X : (y, x) ∈ E(G) \}. \]

Let \( \bar{G} \) denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat \( \bar{G} \) as a directed graph for which the set of its edges is symmetric. Under this convention,

\[ V(\bar{G}) = V(G) \text{ and } E(\bar{G}) = E(G) ∪ E(G^{-1}). \]

A graph S(V(S), E(S)) is called a subgraph of the graph G(V(G), E(G)) if V(S) ⊆ V(G) and E(S) ⊆ E(G).

The essential feature of Jachymski’s work [16] is that the contraction inequality needs to be satisfied only on certain edges of the graph. This opened a new direction in fixed point theory in which a significant amount of works have appeared. Our work is in furtherance of this line of research.

The results obtained in this paper are common fixed point theorems under the assumption of certain almost G-contractions without and with rational terms in metric spaces endowed with the structure of a graph. Each of our results is illustrated with example.

## 2 Main Results

Assume that \((X, d)\) is a metric space, and G is a directed graph such that V(G) = X and Δ ⊆ E(G).

**Definition 2.1.** The triple \((X, d, G)\) is said to be regular if

(i) for any sequence \(\{x_n\}\) in X with \(x_n → x\) and \((x_n, x_{n+1}) ∈ E(G)\) for all \(n ∈ N\), then \((x_n, x) ∈ E(G)\) for all \(n ∈ N\),

(ii) for any sequence \(\{x_n\}\) in X with \(x_n → x\) and \((x_{n+1}, x_n) ∈ E(G)\) for all \(n ∈ N\), then \((x, x_n) ∈ E(G)\) for all \(n ∈ N\).
Theorem 2.2. Let \((X, d)\) be a complete metric space endowed with a directed graph \(G\) and \(S, T : X \to X\) be two mappings. Suppose that (i) \((Sx, TSx)\) and \((Tx, STx)\) \(\in E(G)\) for all \(x \in X\) and (ii) there exist \(\alpha \in (0, 1)\) and \(L \geq 0\) such that for all \(x, y \in X\) with \((x, y) \in E(G)\) or \((x, y) \in E(G^{-1})\),
\[
d(Sx, Ty) \leq \alpha \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\} + L \min\{d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\},
\]
Also suppose that (a) \(S\) or \(T\) is continuous, or (b) the triple \((X, d, G)\) is regular. Then \(S\) and \(T\) have a common fixed point in \(X\).

Proof. First we prove that any fixed point of \(S\) is also a fixed point of \(T\) and conversely, any fixed point of \(T\) is also a fixed point of \(S\).

Now suppose that \(p\) is a fixed point of \(S\) and \(p \neqTp\). Since \((p, p) \in E(G)\), applying the condition (ii), we have
\[
d(p, Tp) = d(Sp, Tp)
\leq \alpha \max\{d(p, p), d(p, Sp), d(p, Tp), \frac{d(p, Tp) + d(p, Sp)}{2}\} + L \min\{d(p, Sp), d(p, Tp), d(p, Tp), d(p, Sp)\}
\leq \alpha \max\{d(p, p), d(p, p), d(p, Tp), \frac{d(p, Tp) + d(p, p)}{2}\} + L \min\{d(p, p), d(p, Tp), d(p, Tp), d(p, p)\}
\leq \alpha d(p, Tp) < d(p, Tp), \quad (\text{since } \alpha < 1),
\]
which is a contradiction. Hence \(p = Tp\), that is, \(p\) is a fixed point of \(T\). Using a similar argument, we have that any fixed point of \(T\) is also a fixed point of \(S\).

Let \(x_0 \in X\). We construct the sequence \(\{x_n\}\) such that
\[
x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \forall n \geq 0. \tag{1}
\]
By the condition (i), \((Sx_0, TSx_0) \in E(G)\), that is, \((x_1, Tx_1) \in E(G)\), that is, \((x_1, x_2) \in E(G)\). Further by the condition (i), \((Tx_1, STx_1) \in E(G)\), that is, \((x_2, Sx_2) \in E(G)\), that is, \((x_2, x_3) \in E(G)\). Again, by the condition (i), \((Sx_2, TSx_2) \in E(G)\), that is, \((x_3, Tx_3) \in E(G)\), that is, \((x_3, x_4) \in E(G)\). Continuing this process we obtain
\[
(x_n, x_{n+1}) \in E(G), \forall n \geq 1. \tag{2}
\]
Since \((x_{2n}, x_{2n+1}) \in E(G)\) for all \(n \geq 1\), applying the condition (ii), we have
\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})
\leq \alpha \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})\} + L \min\{d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\}
\leq \alpha \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),
\]

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Suppose that \( d \) which is a contradiction. Therefore, from (3), we have

\[
\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2} 
+ L \min \left\{ \begin{array}{l} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}) \end{array} \right\} 
\leq \max \left\{ \begin{array}{l} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ \frac{d(x_{2n}, x_{2n+2})}{2} \end{array} \right\}.
\]

As \( \frac{d(x_{2n}, x_{2n+2})}{2} \) \( \leq \max \left\{ \begin{array}{l} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \end{array} \right\} \), it follows from the above inequality that

\[
d(x_{2n+1}, x_{2n+2}) \leq \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \right\}.
\]

(3)

Suppose that \( d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2}). \) Then from (3), we have

\[
d(x_{2n+1}, x_{2n+2}) \leq \alpha \cdot d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2}) \quad \text{(since } \alpha < 1),
\]

which is a contradiction. Therefore, from (3), we have

\[
d(x_{2n+1}, x_{2n+2}) \leq \alpha \cdot d(x_{2n}, x_{2n+1}), \quad \text{for all } n \geq 1.
\]

(4)

By (2), \( (x_{2n+1}, x_{2n+2}) \in E(G) \) and hence \( (x_{2n+2}, x_{2n+1}) \in E(G^{-1}) \) for all \( n \geq 1. \) Applying (ii), we have

\[
d(x_{2n+3}, x_{2n+2}) = d(Sx_{2n+2}, Tx_{2n+1}) 
\leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, Sx_{2n+2}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n+2}, Tx_{2n+1}) + \frac{d(x_{2n+1}, Sx_{2n+2})}{2} \right\} 
+ L \min \left\{ d(x_{2n+2}, Sx_{2n+2}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n+2}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n+2}) \right\}
\leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+2}) + \frac{d(x_{2n+1}, x_{2n+3})}{2} \right\}
+ L \min \left\{ d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+3}) \right\}
\leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), \frac{d(x_{2n+1}, x_{2n+3})}{2} \right\}.
\]

As \( \frac{d(x_{2n+1}, x_{2n+3})}{2} \) \( \leq \max \left\{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}) \right\} \), it follows that

\[
d(x_{2n+2}, x_{2n+3}) \leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}) \right\}.
\]

(5)

Suppose that \( d(x_{2n+1}, x_{2n+2}) < d(x_{2n+2}, x_{2n+3}) \). Then from (5), we have

\[
d(x_{2n+2}, x_{2n+3}) \leq \alpha \cdot d(x_{2n+2}, x_{2n+3}) < d(x_{2n+2}, x_{2n+3}) \quad \text{(since } \alpha < 1),
\]

which is a contradiction. Therefore, from (5), we have

\[
d(x_{2n+2}, x_{2n+3}) \leq \alpha \cdot d(x_{2n+1}, x_{2n+2}), \quad \text{for all } n \geq 1.
\]

(6)

From (4) and (6), we have

\[
d(x_{n+1}, x_{n+2}) \leq \alpha \cdot d(x_{n}, x_{n+1}), \quad \text{for all } n \geq 1.
\]

(7)

Applying (7) repeatedly, we have

\[
d(x_{n}, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_{n}), \leq \alpha^2 \cdot d(x_{n-2}, x_{n-1}), \leq \ldots \leq \alpha^{n-1} \cdot d(x_{1}, x_{2}).
\]

(8)
For any \( m > n \),
\[
d(x_m, x_n) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)
\]
\[
\leq \left[ a^{n-1} + a^n + a^{n+1} + \ldots + a^{m-2} \right] d(x_1, x_2)
\]
\[
\leq \frac{a^{n-1}}{1-a} d(x_1, x_2) \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad (\text{since } 0 < a < 1),
\]
which implies that \( \{x_n\} \) is a Cauchy sequence. From the completeness of \( X \), there exists an \( z \in X \) such that
\[
x_n \longrightarrow z \text{ as } n \longrightarrow \infty. \tag{9}
\]

Suppose that the condition (a) holds.

Let \( S \) be continuous. By (1) and (9), we have
\[
x_{2n+2} = T x_{2n+1} \longrightarrow z \text{ and } x_{2n+3} = S x_{2n+2} \longrightarrow z \text{ as } n \longrightarrow \infty. \tag{10}
\]
Since \( S \) is continuous, using (10) we have
\[
\lim_{n \to \infty} S T x_{2n+1} = S z, \text{ that is, } \lim_{n \to \infty} S x_{2n+2} = S z, \text{ that is, } z = Sz. \tag{11}
\]
Therefore, \( z \) is a fixed point of \( S \). By what we have already proved, \( z \) is also a fixed point of \( T \). Hence \( z \) is a common fixed point of \( S \) and \( T \).

Similarly, if \( T \) is continuous, then \( z \) is a common fixed point of \( S \) and \( T \).

Suppose that the condition (b) holds.

Since \( \{x_n\} \) is a sequence in \( X \) such that \( x_n \longrightarrow z \) and \( (x_n, x_{n+1}) \in E(G) \) for all \( n \geq 1 \), using the regular property of \((X, d, G)\), we have
\[
(x_n, z) \in E(G) \text{ for all } n \geq 1. \tag{12}
\]
Therefore,
\[
(x_{2n}, z) \in E(G) \text{ for all } n \geq 1. \tag{13}
\]
Since \((x_{2n}, z) \in E(G) \) for all \( n \geq 1 \), applying the condition (ii), we have
\[
d(x_{2n+1}, Tz) = d(Sx_{2n}, Tz)
\]
\[
\leq \alpha \max \left\{ d(x_{2n}, z), d(x_{2n}, Sx_{2n}), d(z, Tz), \frac{d(x_{2n}, Tz) + d(z, Sx_{2n})}{2} \right\}
\]
\[
+ L \min \left\{ d(x_{2n}, Sx_{2n}), d(z, Tz), d(x_{2n}, Tz), d(z, Sx_{2n}) \right\}
\]
\[
\leq \alpha \max \left\{ d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz), \frac{d(x_{2n}, Tz) + d(z, x_{2n+1})}{2} \right\}
\]
\[
+ L \min \left\{ d(x_{2n}, x_{2n+1}), d(z, Tz), d(x_{2n}, Tz), d(z, x_{2n+1}) \right\}.
\]
Taking the limit as \( n \longrightarrow \infty \) in the above inequality and using (9), we have
\[
d(z, Tz) \leq \alpha \max \left\{ d(z, z), d(z, z), d(z, Tz), \frac{d(z, Tz) + d(z, z)}{2} \right\}
\]
\[
+ L \min \left\{ d(z, z), d(z, Tz), d(z, Tz), d(z, z) \right\}
\]
\[
\leq \alpha \max \left\{ d(z, Tz), \frac{d(z, Tz)}{2} \right\} \leq a d(z, Tz).
\]
Since \( \alpha < 1 \), the above inequality leads to a contradiction unless \( d(z, Tz) = 0 \), that is, \( z = Tz \), that is, \( z \) is a fixed point of \( T \). By what we have already proved, \( z \) is also a fixed point of \( S \). Hence \( z \) is a common fixed point of \( S \) and \( T \).
In the following theorem mappings are assumed to satisfy a almost $G$-contraction of rational type.

**Theorem 2.3.** Let $(X, d)$ be a complete metric space endowed with a directed graph $G$ and $S, T : X \to X$ be two mappings. Suppose that (i) $(Sx, TSx)$ and $(Tx, STx) \in E(G)$ for all $x \in X$ and (ii) there exist $\alpha \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in X$ with $(x, y) \in E(G)$ or $(x, y) \in E(G^{-1})$,

$$d(Sx, Ty) \leq \alpha \max \left\{ d(x, y), \frac{d(x, y) \left[ 1 + d(x, Ty) d(y, Sx) \right]}{1 + d(x, y)} \right\} + L d(x, Ty) d(y, Sx).$$

Also suppose that (a) $S$ or $T$ is continuous, or (b) the triple $(X, d, G)$ is regular. Then $S$ and $T$ have a common fixed point in $X$.

**Proof.** Like in the proof of Theorem 2.2 we first prove that any fixed point of $S$ is also a fixed point of $T$ and conversely. Now suppose that $p$ is a fixed point of $S$ and $p \neq Tp$. Since $(p, p) \in E(G)$, applying the condition (ii), we have

$$d(p, Tp) = d(Sp, Tp) \leq \alpha \max \left\{ d(p, p), \frac{d(p, p) \left[ 1 + d(p, Tp) d(p, Sp) \right]}{1 + d(p, p)} \right\} + L d(p, Tp) d(p, Sp) \leq L d(p, Tp) d(p, p) = 0,$n

which implies that $d(p, Tp) = 0$, that is, $p = Tp$, that is, $p$ is a fixed point of $T$. Using a similar argument, we have that any fixed point of $T$ is also a fixed point of $S$.

Let $x_0 \in X$. Starting with $x_0$ we construct the same sequence as in the proof of Theorem 2.2. Then the sequence $\{x_n\}$ satisfies (1) and (2), that is,

$$x_{2n+1} = Sx_{2n}$$

and

$$x_{2n+2} = Tx_{2n+1}, \text{ for all } n \geq 0.$$

Since $(x_{2n}, x_{2n+1}) \in E(G)$ for all $n \geq 1$, applying the condition (ii), we have

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq \alpha \max \left\{ d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1}) \left[ 1 + d(x_{2n}, Tx_{2n+1}) d(x_{2n+1}, Sx_{2n+1}) \right]}{1 + d(x_{2n}, x_{2n+1})} \right\} + L d(x_{2n}, Tx_{2n+1}) d(x_{2n+1}, Sx_{2n+1}).$$

By (2), $(x_{2n+1}, x_{2n+2}) \in E(G)$ and hence $(x_{2n+2}, x_{2n+1}) \in E(G^{-1})$ for all $n \geq 1$. Applying (ii), we have

$$d(x_{2n+3}, x_{2n+2}) = d(Sx_{2n+2}, Tx_{2n+1}) \leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+1}) \left[ 1 + d(x_{2n+2}, x_{2n+1}) d(x_{2n+1}, Sx_{2n+2}) \right]}{1 + d(x_{2n+2}, x_{2n+1})} \right\}.$$

Therefore,

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1}), \text{ for all } n \geq 1. \quad (14)$$

By (2), $(x_{2n+1}, x_{2n+2}) \in E(G)$ and hence $(x_{2n+2}, x_{2n+1}) \in E(G^{-1})$ for all $n \geq 1$. Applying (ii), we have

$$d(x_{2n+3}, x_{2n+2}) = d(Sx_{2n+2}, Tx_{2n+1}) \leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+1}) \left[ 1 + d(x_{2n+2}, x_{2n+1}) d(x_{2n+1}, Sx_{2n+2}) \right]}{1 + d(x_{2n+2}, x_{2n+1})} \right\},$$
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\[
\begin{align*}
&\frac{d(x_{2n+2}, x_{2n+1}) \left[ 1 + d(x_{2n+2}, Tz_{2n+1}) d(x_{2n+1}, Sz_{2n+2}) \right]}{1 + d(x_{2n+2}, x_{2n+1})} \\
&\quad + L d(x_{2n+2}, Tz_{2n+1}) d(x_{2n+1}, Sz_{2n+2}) \\
&\leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+1}) \left[ 1 + d(x_{2n+2}, x_{2n+2}) d(x_{2n+1}, x_{2n+3}) \right]}{1 + d(x_{2n+2}, x_{2n+1})} \right. \\
&\quad \left. + L d(x_{2n+2}, x_{2n+2}) d(x_{2n+1}, x_{2n+3}) \right\} \\
&\leq \alpha \max \left\{ d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, x_{2n+1})}{1 + d(x_{2n+2}, x_{2n+1})} \right\} \\
&\leq \alpha d(x_{2n+2}, x_{2n+1}), \quad (\text{since } d(x_{2n+2}, x_{2n+1}) \leq d(x_{2n+2}, x_{2n+1})).
\end{align*}
\]

Therefore,
\[
d(x_{2n+3}, x_{2n+2}) \leq \alpha d(x_{2n+2}, x_{2n+1}), \quad \text{for all } n \geq 1.
\] (15)

From (14) and (15), we have
\[
d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}), \quad \text{for all } n \geq 1.
\] (16)

Arguing similarly as in the proof of Theorem 2.2, we prove that \( \{ x_n \} \) is a Cauchy sequence and there exists \( z \in X \) for which (9) is satisfied, that is, \( x_n \to z \) as \( n \to \infty \).

Suppose that the condition (a) holds.

Arguing similarly as in the proof of the Theorem 2.2 we prove that \( z \) is a common fixed point of \( S \) and \( T \).

Suppose that the condition (b) holds.

Applying similar logic as described in Theorem 2.2, we have \( (x_{2n}, z) \in E(G) \) for all \( n \geq 1 \). Then applying the condition (ii), we have
\[
\begin{align*}
d(x_{2n+1}, Tz) &= d(Sx_{2n}, Tz) \\
&\leq \alpha \max \left\{ d(x_{2n}, z), \frac{d(x_{2n}, z) \left[ 1 + d(x_{2n}, Tz) d(z, Sz_{2n}) \right]}{1 + d(x_{2n}, z)} \right. \\
&\quad \left. + L d(x_{2n}, Tz) d(z, Sz_{2n}) \right\} \\
&\leq \alpha \max \left\{ d(x_{2n}, z), \frac{d(x_{2n}, z) \left[ 1 + d(x_{2n}, Tz) d(z, x_{2n+1}) \right]}{1 + d(x_{2n}, z)} \right. \\
&\quad \left. + L d(x_{2n}, Tz) d(z, x_{2n+1}) \right\}
\end{align*}
\]

Taking the limit as \( n \to \infty \) in the above inequality and using (9), we have \( d(z, Tz) \leq 0 \), which implies that \( d(z, Tz) = 0 \), that is, \( z = Tz \), that is, \( z \) is a fixed point of \( T \). By what we have already proved, \( z \) is also a fixed point of \( S \). Hence \( z \) is a common fixed point of \( S \) and \( T \).

\( \Box \)

**Example 2.4.** Let \( X = A \cup B \), where \( A = [0, 1] \) and \( B = \{ n : n \in \mathbb{N} \text{ and } n \geq 2 \} \). Let \( G \) be a directed graph with \( V(G) = X \) and \( E(G) = \{(x, y) : x, y \in A \text{ and } x \leq y\} \cup \{(n, n) : n \in B\} \). Let \( d \) be the usual metric on \( X \). Let \( S, T : X \to X \) be defined respectively as follows:

\[
Sx = \begin{cases} 
\frac{1}{16}, & \text{if } x \in A \text{ with } 0 \leq x \leq \frac{1}{2}, \\
0, & \text{if } x \in A \text{ with } \frac{1}{2} < x \leq 1, \quad \text{and } \\
x, & \text{if } x \in B,
\end{cases}
\]

\( T \) is defined similarly.

\( (A) \)

Let \( L \geq 0 \) any real number and \( \frac{1}{8} \leq \alpha < 1 \). Then all the conditions of Theorem 2.2 are satisfied and here \( B \cup \{ \frac{1}{16} \} \) is the set of common fixed points of \( S \) and \( T \).
Let $L \geq 256$ any real real number and $\frac{1}{2} \leq \alpha < 1$. Then all the conditions of Theorem 2.3 are satisfied and here $B \cup \{\frac{1}{16}\}$ is the set of common fixed points of $S$ and $T$.

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References