Common best proximity points for generalized $\alpha - \phi$–Geraghty proximal contractions

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Abstract: In this paper, we establish some common best proximity point theorems for generalized $\alpha-\phi$–Geraghty proximal contraction mappings in complete metric spaces. Moreover, we give some examples to illustrate our main results. Our results improve and extend various results given by some authors in literature.

Keywords: Common best proximity point; fixed point; $\alpha$-proximal admissible mapping; $\alpha-\phi$–Geraghty proximal contraction mapping.

MSC: 47H10, 54H25.

1 Introduction

In 1922, Banach [1] proved a theorem, which is well known as “Banach’s fixed point theorem” to establish the existence of solutions for integral equation.
**Theorem 1.1.** Let \((X, d)\) be a complete metric space and \(T : X \to X\) be a contractive mapping (that is, there exists \(L \in [0, 1)\) such that
\[
d(Tx, Ty) \leq Ld(x, y)
\]
for all \(x, y \in X\). Then \(T\) has a unique fixed point \(z \in X\).

Further, since Banach’s fixed point theorem, because of its simplicity, usefulness and applications, it has become a very popular tools solving the existence problems in many branches of mathematics analysis. Many authors have improved, extended and generalized Banach’s theorem in many directions (see in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]) and references therein.

Fixed point theory plays an important role in showing the existence of solutions of various equations of the form \(Tx = x\) for a self-mapping \(T : A \to A\) defined on a subset \(A\) of a metric space \(X\). Now, if \(T : A \to B\) is a non-self mapping, where \(A\) and \(B\) are subsets of \(X\), then the equation \(Tx = x\) does not necessarily have a solution, which is known as a fixed point of the mapping \(T\). Thus, in such circumstance, it may be considered to determine an element \(x\) for which the error \(d(x, Tx)\) is the global minimum, in which case \(x\) and \(Tx\) are close proximity to each other. One of the most interesting results, best approximation theorem, in this direction due to Fan [12].

**Theorem 1.2.** Let \(K\) be a non-empty compact convex subset of a normed space \(X\) and \(T : K \to X\) be a continuous non-self mapping. Then there exists \(x \in K\) such that
\[
\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}.
\]

When a non-self mapping \(T : A \to B\) has not a fixed point, it is quite natural to find an element \(x^*\) such that \(d(x^*, Tx^*)\) is minimum. The best proximity point theorems assure the existence of an element \(x^*\) such that
\[
d(x^*, Tx^*) = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.
\]
This element \(x^*\) is called the **best proximity point** of \(T\). If the mapping \(T\) under discussion is a self-mapping, then the best proximity point theorem becomes to a fixed point results. Afterwards, several authors have been extended this idea in many directions (see in [13, 14, 15, 16, 17]) and references therein.

In 2005, Eldred et al. [18] proved some best proximity point theorems for relatively non-expansive mappings in metric spaces. Since then, in 2011, Basha [19] defined proximal contractions of the first and second kinds in metric spaces. Moreover, he gave a necessary and sufficient condition to show the existence of a best proximity point for proximal contraction of first kind and the second kind which are non-self mapping analogues of contraction self-mappings and also established some best proximity theorems and convergence theorems.

On the other hand, one of interesting generalized Banach’s fixed point theorem is due to Geraghty [20]. We introduce the class \(\mathcal{F}\) of those function \(\beta : [0, \infty) \to [0, 1)\) satisfying the following condition:
\[
\beta(t_n) \to 1 \implies t_n \to 0.
\]
Theorem 1.3. Let \((X,d)\) be a complete metric space and \(T : X \to X\) be a self-mapping. Suppose that there exists \(\beta \in \mathcal{F}\) such that, for any \(x,y \in X\),
\[
d(Tx, Ty) \leq \beta(d(x, y))d(x, y).
\]
Then \(T\) has a unique fixed point.

Since the constant function \(f(t) = k\), where \(k \in [0, 1)\), in \(\mathcal{F}\), Theorem 1.3 extends Theorem 1.1.

In 2014, Karapinar [21] introduced a generalized \(a\)-\(\varphi\)-Geraghty contraction type mapping and, in 2016, Hamzehnejadi and Lashkaripour [22] introduced a generalized \(a\)-\(\varphi\)-Geraghty proximal contraction mapping. They also established some best proximity point theorems for this mapping.

Now, we introduce the class \(\Phi\) of all functions \(\varphi : [0, \infty) \to [0, \infty)\) satisfying the follows conditions:

(i) \(\varphi\) is nondecreasing;
(ii) \(\varphi\) is continuous;
(iii) \(\varphi(t) = 0\) if and only if \(t = 0\).

Definition 1.4. Let \(A, B\) be nonempty subsets of a metric space \((X,d)\) and \(a : X \times X \to [0, \infty)\) be a function. A mapping \(T : A \to B\) is called a generalized \(a\)-\(\varphi\)-Geraghty proximal contraction if there exits \(\beta \in \mathcal{F}\) such that
\[
\begin{align*}
d(u, Tx) &= d(A, B) \\
d(v, Ty) &= d(A, B)
\end{align*}
\]
implies \(a(x, y)\varphi(d(u, v)) \leq \beta(\varphi(M(x, y, u, v)))\varphi(M(x, y, u, v))\),
where
\[
M(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v)\}
\]
for all \(x, y, u, v \in A\) and \(\varphi \in \Phi\).

Recently, Aydi et al. [23] gives some results on common best proximity points for \(a\)-\(\varphi\)-proximal contractive mappings in metric spaces.

Definition 1.5. Let \(A\) and \(B\) be nonempty subsets of a metric space \((X,d)\). Let \(\psi \in \Psi\) and \(a : X \times X \to [0, \infty)\) be a function. Let \(S, T : A \to B\) be non-self mappings.

(1) \((S, T)\) is called a generalized \(a\)-\(\varphi\)-proximal contraction pair if
\[
d(Sx, Ty) \leq \psi(M(x, y))
\]
for all \(x, y \in A\) with \(a(x, y) \geq 1\), where
\[
M(x, y) = \max\left\{d(x, y), \frac{d(x, Sx) + d(y, Ty) - 2d(A, B)}{2}, \frac{d(y, Sx) + d(x, Ty) - 2d(A, B)}{2}\right\}
\]
(2) \((S, T)\) is called a generalized \(a\)-\(\varphi\)-proximal contraction pair of the first kind if
\[
\begin{align*}
a(x, y) &\geq 1 \\
d(u, Sx) &= d(A, B) \implies d(u, v) \leq \psi(M(x, y)) \\
d(v, Ty) &= d(A, B)
\end{align*}
\]
for all \(x, y, u, v \in A\).
Motivation from Definitions 1.4 and 1.5, we combine two definitions and establish common best proximity points theorems for a generalized $\alpha$-$\phi$-Geragthy contraction mappings in metric spaces. Moreover, we give some examples and applications to illustrate the main results in this paper.

2 Preliminaries

Now, we recall some elementary results and basic definitions for our main results in this paper.

Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$.

$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$

$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\},$

$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$

**Definition 2.1.** Let $(X, d)$ be a metric space and $A, B$ two nonempty subsets of $X$. A point $u \in X$ is called a **common best proximity point** of non-self mappings $S, T : A \to B$ if

$$d(u, Su) = d(u, Tu) = d(A, B).$$

It is clear that a common fixed point coincides with a common best proximity point if $d(A, B) = 0$.

**Definition 2.2.** [24]. Let $A, B$ be nonempty subsets of a metric space $(X, d)$ and $\alpha : X \times X \to [0, \infty)$ be a function. A pair $(S, T)$ of non-self mappings $S, T : A \to B$ is said to be **$\alpha$-proximal admissible** if

$$\begin{align*}
\alpha(x_1, x_2) &\geq 1 \\
\min\{d(u_1, Sx_1), d(u_2, T\alpha x_2)\} &\geq 1
\end{align*}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Now, we introduce a generalized $\alpha$-$\phi$-Geragthy proximal contraction mappings in metric spaces.

**Definition 2.3.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. Let $\phi \in \Phi, \alpha : X \times X \to [0, \infty)$ be a function $S, T : A \to B$ be non-self mappings. $(S, T)$ is called a **generalized $\alpha$-$\phi$-Geragthy proximal contraction pair** if

$$\begin{align*}
\alpha(x, y) &\geq 1 \\
\alpha(x, y)\phi(d(u, v)) &\leq \beta(\phi(M(x, y)))\phi(M(x, y) - d(A, B))
\end{align*}$$

for all $x, y, u, v \in A$ with $\alpha(x, y) \geq 1$, where $\beta \in \mathcal{F}$ and

$$M(x, y) = \max\{d(x, y), d(x, u), d(y, v)\}. \quad (2.1)$$
3 Main results

Now, we prove some common best proximity point theorems for a generalized $\alpha$-$\phi$-Geragthy proximal contraction pair in metric spaces.

**Theorem 3.1.** Let $A$ and $B$ be nonempty subsets of a complete metric space $(X,d)$ such that $A_0 \neq \emptyset$ and $A$ is closed. Let $S, T : A \to B$ be a generalized $\alpha$-$\phi$-Geragthy proximal contraction pair. Suppose that

(i) $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$;

(ii) there exists $x_0, x_1 \in A_0$ such that
\[ d(x_1, Sx_0) = d(A, B), \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1; \]

(iii) the pair $(S, T)$ is $\alpha$-proximal admissible;

(iv) $S$ and $T$ are continuous.

Then $S$ and $T$ has a common best proximity point, that is, there exists a point $x^* \in A$ such that
\[ d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B). \]

**Proof.** By the condition (ii), there exist $x_0, x_1 \in A_0$ such that
\[ d(x_1, Sx_0) = d(A, B), \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1. \tag{3.1} \]

From the condition (i), we have $Tx_1 \in B_0$ and so there exists $x_2 \in A_0$ such that
\[ d(x_2, Tx_1) = d(A, B). \tag{3.2} \]

From (3.1) and (3.2), since the pair $(S, T)$ is $\alpha$-proximal admissible, we obtain
\[ \min\{\alpha(x_1, x_2), \alpha(x_2, x_1)\} \geq 1. \tag{3.3} \]

Again, by the condition (i), we have $Sx_2 \in B_0$ and so there exists $x_3 \in A_0$ such that
\[ d(x_3, Sx_2) = d(A, B). \tag{3.4} \]

By induction, we can find the sequence $\{x_n\}$ in $A_0$ such that
\[ \min\{\alpha(x_n, x_{n+1}), \alpha(x_{n+1}, x_n)\} \geq 1 \tag{3.5} \]

and
\[ d(x_{n+1}, Sx_n) = d(x_{n+2}, Tx_{n+1}) = d(A, B) \tag{3.6} \]

for all $n \geq 0$. Since $(S, T)$ is a generalized $\alpha$-$\phi$-Geragthy proximal contraction pair, using (3.5) and $\alpha(x_n, x_{n+1}) \geq 1$, it follows that, for all $n \geq 0$,
\[ \phi(d(x_{n+1}, x_{n+2})) \leq \alpha(x_n, x_{n+1})\phi(d(x_{n+1}, x_{n+2})) \leq \beta(\phi(M(x_n, x_{n+1})))\phi(M(x_n, x_{n+1}) - d(A, B)) \]
for all \( n \) that \( n \) such that \( d(x_n, x_{n+1}) = 0 \). Suppose that \( d(x_{n_0}, x_{n_0+1}) = 0 \). This implies that \( x_{n_0} = x_{n_0+1} = x_{n_0+2} \). By (3.6), we obtain
\[
d(x_{n_0}, Sx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = d(A, B),
\]
which is the desired result.

Now, let \( r = d(x_n, x_{n+1}) \neq 0 \) for all \( n \geq 0 \). In the sequel, we prove that \( r = 0 \). Suppose that \( r > 0 \). From (3.7) and (3.9), we have
\[
0 < \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}) - d(A, B))} \leq \beta(\phi(d(x_n, x_{n+1}))).
\]
Using the fact that $\phi$ is non-decreasing, we have
\[ 0 < \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}))} \leq \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}) - d(A, B))} \leq \beta(\phi(d(x_n, x_{n+1}))), \]
which implies that $\lim_{n \to \infty} \beta(\phi(d(x_n, x_{n+1}))) = 1$. By the property of $\beta \in I$, we have
\[ \lim_{n \to \infty} \phi(d(x_n, x_{n+1})) = 0. \]
Hence we get $r = 0$, which is a contradiction. Therefore, $\lim_{n \to \infty} d(x_{n+1}, x_{n+2}) = 0$. Note that, for all $m, n \geq 0$,
\[ \min\{\alpha(x_n, x_m), \alpha(x_m, x_n)\} \geq 1 \quad (3.11) \]
and
\[ d(x_{n+1}, Sx_n) = d(x_{m+1}, T_xm) = d(A, B). \quad (3.12) \]

Thus, for all $m, n \geq 0$, we obtain
\[ \phi(d(x_{n+1}, x_{m+1})) \leq \alpha(x_n, x_m)\phi(d(x_{n+1}, x_{m+1})) \leq \beta(\phi(M(x_{n+1}, x_{m+1})))\phi(M(x_{n+1}, x_{m+1}) - d(A, B)), \quad (3.13) \]
where
\[ M(x_{n+1}, x_{m+1}) = \max\{d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1})\} \leq \max\{d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1})\} + d(A, B). \]

Since $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$, we have
\[ \limsup_{m,n \to \infty} M(x_{n+1}, x_{m+1}) = \limsup_{m,n \to \infty} d(x_n, x_m) \leq \limsup_{m,n \to \infty} d(x_n, x_m) + d(A, B). \quad (3.14) \]

Now, we show that $\{x_n\}$ is a Cauchy sequence. In the contrary, we suppose that
\[ \limsup_{m,n \to \infty} d(x_n, x_m) = r > 0. \]

Letting $m, n \to \infty$ and using the triangular inequality, we derive
\[ \limsup_{m,n \to \infty} d(x_n, x_m) \leq \limsup_{m,n \to \infty} (d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m)) \leq \limsup_{m,n \to \infty} d(x_{n+1}, x_{m+1}). \quad (3.15) \]

Combining (3.13), (3.14) and (3.15) with the continuous property of $\phi$, we obtain
\[ \limsup_{m,n \to \infty} \phi(d(x_n, x_m)) \leq \limsup_{m,n \to \infty} \beta(\phi(M(x_{n+1}, x_{m+1}))) \limsup_{m,n \to \infty} \phi(d(x_n, x_m)). \]
Since $\limsup_{m,n \to \infty} d(x_n, x_m) = r > 0$, it follows that
\[ \limsup_{m,n \to \infty} \beta(\phi(M(x_{n+1}, x_{m+1}))) = 1. \]
Since $\beta \in I$, we have
\[ \limsup_{m,n \to \infty} d(x_n, x_m) = \limsup_{m,n \to \infty} M(x_{n+1}, x_{m+1}) = 0, \]
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which is a contradiction. Therefore, \( \{x_n\} \) is a Cauchy sequence in \( A \). Since \( A \) is a closed subset of a complete metric space \( (X, d) \), there exists \( x^* \in A \) such that \( x_n \to x^* \) as \( n \to \infty \). Since \( S \) is continuous at \( x^* \),
\[
\lim_{n \to \infty} Sx_n = Sx^*.
\]
Moreover, from the continuous property of the metric \( d \), it follows that \( \lim_{n \to \infty} d(x_{n+1}, Sx_n) = d(x^*, Sx^*) \).

Following the proof of Theorem 3.1, we obtain a sequence

\[
\{x_n\} \rightarrow x^* \quad \text{as} \quad n \rightarrow \infty.
\]

Proof.

\textbf{Theorem 3.2.} Let \( A \) and \( B \) be nonempty subsets of a complete metric space \( (X, d) \) such that \( A \neq \emptyset \) and \( A \) is closed. Let \( S, T : A \to B \) be a generalized \( \alpha \)-\( \phi \)-Geraghty proximal contraction pair. Suppose that

\begin{enumerate}[(i)]
  \item \( S(A_0) \subseteq B_0 \) and \( T(A_0) \subseteq B_0 \);
  \item there exist \( x_0, x_1 \in A_0 \) such that
    \[
    d(x_1, Sx_0) = d(A, B), \quad \min\{a(x_0, x_1), a(x_1, x_0)\} \geq 1;
    \]
  \item the pair \( (S, T) \) is \( \alpha \)-proximal admissible;
  \item the condition (H) holds.
\end{enumerate}

Then there exists \( x^* \in A \) such that \( d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B) \).

\textit{Proof.} Following the proof of Theorem 3.1, we obtain a sequence \( \{x_n\} \) in \( A_0 \) such that (3.5) and (3.6) hold. Also, we can show that \( \{x_n\} \) is a Cauchy sequence in \( A \). Since \( A \) is a closed subset of a complete metric space \( (X, d) \), there exists \( x^* \in A \) such that \( x_n \to x^* \) as \( n \to \infty \). By the condition (H), we have
\[
\alpha(x^*, x_{2n(k)+1}) \geq 1.
\]
Next, from (3.6), it follows that
\[
\quad \quad \quad d(x_{2n(k)+1}, Sx^*) = d(x_{2n(k)+2}, Tx_{2n(k)+1}) = d(A, B).
\] (3.16)

On the other hand, by using the triangular inequality, we obtain
\[
\begin{align*}
d(x^*, Sx^*) & \leq d(x^*, x_{2n(k)+2}) + d(x_{2n(k)+2}, x_{2n(k)+1}) + d(x_{2n(k)+1}, Sx^*) \\
& \leq d(x^*, x_{2n(k)+2}) + d(x_{2n(k)+2}, x_{2n(k)+1}) + d(A, B),
\end{align*}
\]
which implies that
\[
d(x^*, Sx^*) - d(x^*, x_{2n(k)+2}) - d(A, B) \leq d(x_{2n(k)+2}, x_{2n(k)+1}).
\]
Since \( (S, T) \) is a generalized \( \alpha \)-\( \phi \)-Geraghty proximal contraction pair, using the property of \( \phi \) and \( \alpha(x^*, x_{2n(k)+1}) \geq 1 \), we have
\[
\phi(d(x^*, Sx^*)) - d(x^*, x_{2n(k)+2}) - d(A, B))
\]
where
\[ M(x^*, x_{2n(k)+1}) = \max\{d(x^*, x_{2n(k)+1}), d(x^*, x_{2n(k)+2})\} \]
\[ = \max\{d(x^*, x_{2n(k)+1}), d(x_{2n(k)+1}, x_{2n(k)+2})\} \]
\[ \leq \max\{d(x^*, x_{2n(k)+1}), d(x_{2n(k)+1}, x_{2n(k)+2})\} + d(A, B). \]

Observe that
\[ \lim_{k \to \infty} d(x^*, x_{2n(k)+1}) = \lim_{k \to \infty} d(x_{2n(k)+1}, x_{2n(k)+2}) = \lim_{k \to \infty} d(x^*, x_{2n(k)+2}) = 0. \]
Thus there exists \( N \in \mathbb{N} \) such that, for all \( k \geq N \),
\[ M(x^*, x_{2n(k)+1}) = 0 \leq d(A, B). \] (3.18)

From (3.17), (3.18) and the property of \( \phi \), it follows that, for all \( k \geq N \),
\[ d(x^*, Sx^*) - d(x^*, x_{2n(k)+2}) - d(A, B) \leq M(x^*, x_{2n(k)+1}) - d(A, B) \]
\[ \leq d(A, B) - d(A, B) \]
\[ \leq 0 \]
and so
\[ d(x^*, Sx^*) \leq d(x^*, x_{2n(k)+2}) + d(A, B). \]
Letting \( k \to \infty \), we obtain \( d(x^*, Sx^*) = d(A, B). \) Similarly, we have \( d(x^*, Tx^*) = d(A, B). \) Therefore, \( x^* \) is a common best proximity point of \( S \) and \( T \). This completes the proof.

Now, we prove the uniqueness of such a common best proximity point as in Theorems 3.1 and 3.2. Here, we need the following additional condition:

(U): For all \( x, y \in CB(S, T) \), we have \( a(x, y) \geq 1 \), where \( CB(S, T) \) denotes the set of common best proximity points of \( S \) and \( T \).

**Theorem 3.3.** Adding the condition (U) to the hypothesis of Theorem 3.1 (resp., Theorem 3.2), the point \( x^* \) is the unique common best proximity point of \( S \) and \( T \).

**Proof.** Suppose that there exist \( x^*, y^* \in A \) such that
\[ d(x^*, Sx^*) = d(x^*, Tx^*) = d(y^*, Sy^*) = d(y^*, Ty^*) = d(A, B), \]
where \( x^* \neq y^* \). By the condition (U), we have \( a(x^*, y^*) \geq 1 \) and, since \((S, T)\) is a generalized \( \alpha \)-\( \phi \)-Geraghty proximal contraction pair, we have
\[ \phi(d(x^*, y^*)) \leq a(x^*, y^*)\phi(d(x^*, y^*)) \]
\[
\phi(M(x^*, y^*)) \leq \beta(\phi(M(x^*, y^*))) \phi(M(x^*, y^*) - d(A, B)) \\
\phi(M(x^*, y^*)) < \phi(M(x^*, y^*) - d(A, B)),
\]

(3.19)

where

\[
M(x^*, y^*) = \max\{d(x^*, y^*), d(x^*, x^*), d(y^*, y^*)\} \\
= d(x^*, y^*).
\]

From (3.19), we obtain

\[
\phi(d(x^*, y^*)) < \phi(d(x^*, y^*) - d(A, B)) \\
< \phi(d(x^*, y^*)),
\]

which is a contradiction. Hence \( x^* = y^* \). \( \square \)

Next, we give an example to illustrate Theorem 3.1.

**Example 3.4.** Let \( X = [0, \infty) \times [0, \infty) \) be endowed with the metric

\[ d((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + |x_2 - y_2|. \]

Take \( A = \{0\} \times [0, 5] \) and \( B = \{1\} \times [0, 5] \). We know that \( d(A, B) = 1 \), \( A_0 = A \) and \( B_0 = B \). Consider the mappings \( S, T : A \to B \) defined by

\[
S(0, x) = \left(1, \frac{4}{5} \ln \left(1 + \frac{x}{2}\right)\right), \quad T(0, x) = \left(1, \frac{4}{5} \ln(1 + x)\right),
\]

respectively. Then we have \( S(A_0) \subseteq B_0 \) and \( T(A_0) \subseteq B_0 \). Also, define a function \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha((x, y), (x', y')) = 1 \quad \text{if} \quad (x, y), (x', y') \in [0, 1] \times [0, 1], \\
\alpha((x, y), (x', y')) = 0 \quad \text{otherwise}.
\]

Let \( (0, x_1), (0, x_2), (0, u_1) \) and \( (0, u_2) \) in \( A \) such that

\[
\alpha((0, x_1), (0, x_2)) \geq 1, \\
d((0, u_1), S(0, x_1)) = d(A, B), \\
d((0, u_2), T(0, x_2)) = d(A, B).
\]

Then we have \( x_1, x_2 \in [0, 1] \times [0, 1] \). Also, we have \( u_1 = \frac{4}{5} \ln(1 + x_2) \) and \( u_2 = \frac{4}{5} \ln(1 + x_2) \), which implies that

\[
\min\{\alpha((0, u_1), (0, u_2)), \alpha((0, u_2), (0, u_1))\} \geq 1.
\]

Thus the pair \( (S, T) \) is \( \epsilon \)-proximal admissible.

Now, we check that \( (S, T) \) is a generalized \( \epsilon \)-Geraghty proximal contraction pair. Define the functions \( \phi : [0, \infty) \to [0, \infty) \) and \( \beta : [0, \infty) \to [0, 1) \) by

\[
\phi(t) = \frac{\sqrt{2}}{2}, \quad \beta(t) = \frac{\arctan(t)}{t}
\]
for all \( t \geq 0 \), respectively. Then \( \beta \in \mathcal{F} \) and \( \phi \in \Phi \). Let \((x, y) \in A\). Then \( t = d(x, y) = [0, 1] \). Also, it easy to show that
\[
\frac{1}{2} \left( \frac{4}{5} \ln(1 + t) \right)^3 \leq \arctan \left( \frac{t^3}{2} \right)
\] (3.20)
for all \( t \in [0, 1] \). Moreover, we have
\[
\alpha((0, x_1), (0, x_2)) \phi(d(0, u_1), (0, u_2)) = \frac{1}{2} \left| \left| u_1 - u_2 \right| \right|^3
\]
\[
= \frac{1}{2} \left( \left| \frac{4}{5} \ln \left( 1 + \frac{x_1}{2} \right) - \frac{4}{5} \ln \left( 1 + \frac{x_2}{2} \right) \right| \right)^3
\]
\[
= \frac{1}{2} \left( \left| \frac{4}{5} \ln \left( \frac{1 + \frac{x_1}{2}}{1 + \frac{x_2}{2}} \right) \right| \right)^3
\]
\[
\leq \frac{1}{2} \left( \left| \frac{4}{5} \ln \left( \frac{1 + \frac{x_1}{2}}{1 + \frac{x_2}{2}} \right) \right| \right)^3.
\] (3.21)

Now, we show that
\[
\left| \ln \left( \frac{1 + \frac{x_1}{2}}{1 + \frac{x_2}{2}} \right) \right| \leq \ln \left( 1 + \left| \frac{x_1}{2} - \frac{x_2}{2} \right| \right) \leq \ln(1 + |x_1 - x_2|).
\]

Suppose that \( x_1 > x_2 \) or \( x_2 > x_1 \). Observe that
\[
\ln \left( \frac{1 + \frac{x_1}{2}}{1 + \frac{x_2}{2}} \right) \leq \ln \left( \frac{1 + \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_2}{2}}{1 + \frac{x_2}{2}} \right)
\]
\[
\leq \ln \left( 1 + \frac{x_1}{2} - \frac{x_2}{2} \right)
\]
\[
\leq \ln \left( 1 + \frac{x_1}{2} - \frac{x_2}{2} \right)
\]
\[
\leq \ln(1 + |x_1 - x_2|).
\]

Form (3.21), it follows that
\[
\alpha((0, x_1), (0, x_2)) \phi(d(0, u_1), (0, u_2)) \leq \frac{1}{2} \left( \frac{4}{5} \ln(1 + |x_1 - x_2|) \right)^3
\]
\[
\leq \arctan \left( \frac{1}{2} (|x_1 - x_2|)^3 \right)
\]
\[
\leq \arctan \left( \frac{1}{2} (d(0, x_1), (0, x_2))^3 \right).
\] (3.22)

Consider
\[
M(x_1, x_2) = \max \{d(0, x_1), (0, x_2), d(0, x_1), (0, u_1), d(0, x_2), (0, u_2)\}
\]
\[
= \max \{d(0, x_1), (0, x_2), d(0, x_1), (0, u_1), d(0, x_2), (0, u_2)\} + d(A, B).
\]

Taking \( M^*(x_1, x_2) = M(x_1, x_2) + d(A, B) \), it follows form (3.22) that
\[
\alpha((0, x_1), (0, x_2)) \phi(d(0, u_1), (0, u_2)) \leq \arctan \left( \frac{1}{2} (d(0, x_1), (0, x_2))^3 \right)
\]
\[
\leq \arctan \left( \frac{1}{2} (M(x_1, x_2))^3 \right)
\]
\[
\leq \arctan \left( \frac{1}{2} (M(x_1, x_2))^3 \right)
\]
\[
\leq \arctan \left( \frac{1}{2} (M(x_1, x_2))^3 \right) \frac{1}{2} (d(0, x_1), (0, x_2))^3
\]
\[
\leq \frac{1}{2} (d(0, x_1), (0, x_2))^3.
\]
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\[
= \arctan \left( \frac{1}{2} \left( M(x_1, x_2) \right)^3 \right) \frac{1}{1 - \left( \frac{1}{2} \left( M(x_1, x_2) \right)^3 \right)}.
\]

Since \( d(A, B) = 1 \) and \( M^*(x_1, x_2) = M(x_1, x_2) + d(A, B) \), we have

\[
a((0, x_1), (0, x_2)) \phi(0, u_1), (0, u_2)) \leq \arctan \left( \frac{1}{2} \left( M(x_1, x_2) \right)^3 \right) \frac{1}{1 - \left( \frac{1}{2} \left( M(x_1, x_2) \right)^3 \right)} \frac{1}{2} (M(x_1, x_2) + d(A, B) - d(A, B))^3
\]

\[
\leq \arctan \left( \frac{1}{2} \left( M(x_1, x_2) \right)^3 \right) \frac{1}{2} (M^*(x_1, x_2) - d(A, B))^3.
\]

Hence, we have

\[
a((0, x_1), (0, x_2)) \phi((0, u_1), (0, u_2)) \leq \beta(\phi(M(x_1, x_2))) \phi(M^*(x_1, x_2) - d(A, B))
\]

and so \((S, T)\) is a generalized \(\alpha\)-\(\phi\)-Geraghty proximal contraction pair. Furthermore, \(S\) and \(T\) are continuous. Moreover, the condition (ii) of Theorem 3.1 is verified. Indeed, for \(x_0 = (0, 2)\) and \(x_1 = (0, 0.5545)\), we get

\[
d(x_1, Sx_0) = d((0, 0.5545), (1, 0.5545)) = 1 = d(A, B)
\]

and

\[
\min \{a(x_0, x_1), a(x_1, x_0)\} \geq 1.
\]

Hence all the hypothesis of Theorem 3.1 are verified. So, the pair \((S, T)\) admits a common best proximity point, which is \(x^* = (0, 0)\). It is easy to show that the common best proximity point \(x^* = (0, 0)\) is unique.

4 Applications

As consequences of our results, we have the following some applications:

4.1. Some classical best proximity point results

We have the following results from Theorems 3.1 and 3.2.

**Corollary 4.1.** Let \(A\) and \(B\) be nonempty subsets of a complete metric space \((X, d)\) such that \(A_0 \neq \emptyset\) and \(A\) is closed. Let \(S, T : A \rightarrow B\) be a generalized \(\alpha\)-\(\phi\)-Geraghty proximal contraction pair. Suppose that

(i) \(S(A_0) \subseteq B_0\) and \(T(A_0) \subseteq B_0\);

(ii) there exist \(x_0, x_1 \in A_0\) such that

\[
d(x_1, Sx_0) = d(A, B), \quad \min \{a(x_0, x_1), a(x_1, x_0)\} \geq 1;
\]

(iii) the pair \((S, T)\) is \(\alpha\)-proximal admissible;

(iv) \(S\) and \(T\) are continuous or the condition (H) holds.

Then \(S\) and \(T\) has a common best proximity point.
Proof. The proof is similar to that of Theorem 3.1 when $T$ is continuous or that of Theorem 3.2 when the condition (H) holds.

Corollary 4.2. Let $A$ and $B$ be nonempty subsets of a complete metric space $(X, d)$ such that $A_0 \neq \emptyset$ and $A$ is closed. Let $S, T : A \to B$ be non-self mappings, $\alpha : X \times X \to [0, \infty)$ be a function and let $\phi \in \Phi$. Suppose that there exists $k \in [0, 1)$ such that

$$\begin{cases} 
\alpha(x, y) \geq 1 \\
\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
d(u, Sx) = d(A, B) \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
d(v, Ty) = d(A, B) 
\end{cases}$$

for all $x, y \in A$ with $\alpha(x, y) \geq 1$, where $M(x, y)$ is defined by (2.1). Also, assume that

(i) $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$;

(ii) there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Sx_0) = d(A, B), \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iii) the pair $(S, T)$ is $\alpha$-proximal admissible;

(iv) $S$ and $T$ are continuous or the condition (H) holds.

Then $S$ and $T$ has a common best proximity point.

Proof. Letting $\beta(\phi(M(x, y))) = k$. Since $\phi(M(x, y) - d(A, B)) \leq \phi(M(x, y))$, the proof is similar to that of Theorem 3.1 when $T$ is continuous or that of Theorem 3.2 when the condition (H) holds.

Corollary 4.3. Let $A$ and $B$ be nonempty subsets of a complete metric space $(X, d)$ such that $A_0 \neq \emptyset$ and $A$ is closed. Let $S, T : A \to B$ be non-self mappings, $\alpha : X \times X \to [0, \infty)$ be a function and let $\phi \in \Phi$. Suppose that there exists $k \in [0, 1)$ such that

$$\begin{cases} 
\alpha(x, y) \geq 1 \\
\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
d(u, Sx) = d(A, B) \quad \Rightarrow \quad \alpha(x, y) \phi(d(u, v)) \leq k \phi(d(x, y)) \\
d(v, Ty) = d(A, B) 
\end{cases}$$

for all $x, y \in A$ with $\alpha(x, y) \geq 1$. Also, assume that

(i) $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$;

(ii) there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Sx_0) = d(A, B), \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iii) the pair $(S, T)$ is $\alpha$-proximal admissible;

(iv) $S$ and $T$ are continuous or the condition (H) holds.

Then $S$ and $T$ has a common best proximity point.
Proof. Letting \( \beta(\phi(M(x,y))) = k \) and \( M(x,y) = d(x,y) \). Since \( \phi(M(x,y) - d(A,B)) = \phi(d(x,y) - d(A,B)) \leq \phi(d(x,y)) \), the proof is similar to that of Theorem 3.1 when \( T \) is continuous or that of Theorem 3.2 when the condition (H) holds.

\[ \]
(i) $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$;

(ii) there exist $x_0, x_1 \in A_0$ such that
$$d(x_1, Sx_0) = d(A, B), \quad (x_0, x_1), (x_1, x_0) \in E(G);$$

(iii) the pair $S, T$ is $G$-proximal;

(iv) $T$ is continuous or the condition $(H_G)$ holds.

Then there exists $x^* \in A$ such that
$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B).$$

Proof. It suffices to consider a function $\alpha : X \times X \to [0, \infty)$ defined by
$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G), \\ 0 & \text{otherwise}. \end{cases}$$

All the conditions of Theorem (3.1) (resp., Theorem 3.2) are satisfied. This completes the proof. \hfill \Box

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